A *J*-Spectral Factorization Approach to H_∞ Control Problems

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A J-SPECTRAL FACTORIZATION APPROACH TO H_∞ CONTROL PROBLEMS

PROEFSCHRIFT

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prof.dr. A. Bagchi

en de assistent-promotor
dr. H.J. Zwart

Preface

Almost four years have gone since the day I came to The Netherlands to join the Systems, Signals and Control group of the Faculty of Applied Mathematics, University of Twente, and I would like to thank prof. Arun Bagchi for the opportunity he offered me.

After about a year, I chose the topic for my research towards a PhD degree: control theory for infinite-dimensional systems. In the last three years I have managed to write this thesis. A very special word of thanks goes to Hans Zwart, my daily supervisor. I thank him for the constant guidance he gave me during the last three years. The door of his office was always open for me, and he has always found time to answer my questions and to correct my mistakes. I could have never written this thesis without the knowledge I have got from him.

I would like to thank Goran, my "best officemate", for the pleasant time we had. I have shared with him not only the joy of my achievements, but also the frustrations of a young scientist.

I would like to express my gratitude to prof. Ruth Curtain for the useful discussions we had, and for realizing a possible connection of my research with the work of Amol. I thank both for the very pleasant and fruitful time I spent in Groningen during one week visit, in 2000. I would like also to thank prof. Rien Kaashoek for the opportunity to give a talk at the Vrije Universiteit in Amsterdam, and for his useful comments.

I thank all members of the Systems, Signals and Control group for the pleasant scientific and social environment. With the help of Hans Zwart, Jan Willem Polderman and Gjerit Meinsma, I have got addicted to using LaTeX code, and Linux as the main operating system.

I am indebted to professors V. Barbu, R. Curtain, M.A. Kaashoek, G. Meinsma, A. van der Schaft and M. Verhaegen for taking time to be in my promotion committee, and to read the draft version of my thesis.

Thanks to all my friends in Enschede, Romania and elsewhere. Special thanks to Adriana, Maria, Rita, Alex, Edwin and Octav.

I thank my parents and Maria for their support.

Contents

Chapter 1

Introduction

Systems and control theory is an area of research which combines engineering with mathematics. Using mathematical concepts, many engineering problems can be formulated into a mathematical framework and then be solved. One of the significant accomplishments in the field of system and control theory has been the development of the theory of H_∞ control. H_∞ is one member of the family of spaces introduced by the mathematician Hardy. It is the space of functions on the complex plane that are analytic and bounded in the open right half-plane, $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$.

H_∞ optimization of control systems deals with the minimization of the peak value of certain closed-loop frequency response functions. To clarify this, consider as an example the basic feedback system of Figure 1.1. The plant has the trans-

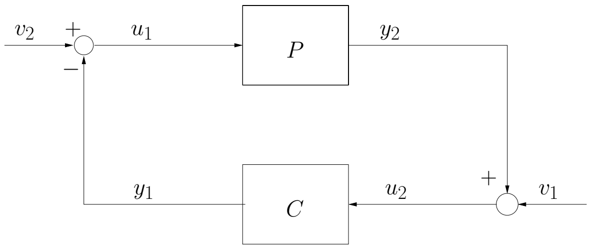

Figure 1.1: Basic feedback system.

fer function P and the compensator has the transfer function C. The signal v_1 represents a disturbance acting on the output of the plant, and the signal v_2 the disturbance acting on the input of the plant. We say that C stabilizes the plant P if the signals u_1 and u_2 are square integrable (have finite energy) whenever v_1 and v_2 are square integrable. The output sensitivity matrix S is defined as being the transfer function from v_1 to u_2 and is given by

$$S = (I + PC)^{-1}.$$

The effects of the disturbance v_1 on the plant output can be made "small" by making the output sensitivity function "small" (ideally, $S = 0$). The problem (originally considered by Zames [80, 81]) is that of finding a compensator C that makes the closed-loop system stable and minimizes the peak value of the output sensitivity function. This peak value is defined as

$$\|S\|_\infty := \sup_{s \in \mathbb{C}_0} |S(s)|,$$

where $\mathbb{C}_0$ denotes the imaginary axis. The justification of this problem is that if the peak value $\|S\|_\infty$ of the sensitivity function is small, then the magnitude of S is necessarily small for all frequencies, so that disturbances are uniformly attenuated over all frequencies. Minimization of $\|S\|_\infty$ is an example of worst-case optimization, because it amounts to minimizing the effect on the output on the worst disturbance (namely, a harmonic disturbance at the frequency where $|S|$ has its peak value).

The worst-case model has another important mathematical interpretation. Suppose that the disturbance v_1 has unknown frequency content, but finite energy

$$\|v_1\|_2^2 := \int_0^\infty |v_1(t)|^2 dt < \infty,$$

where $\|\cdot\|_2$ is the norm in L_2, the space of square integrable functions. By stability we know that u_2 has finite energy. Furthermore, using Paley-Wiener results, the induced operator norm $\|S\|$, defined as

$$\|S\| := \sup_{v_1 \in L_2} \frac{\|u_2\|_2}{\|v_1\|_2},$$

is nothing other than $\|S\|_\infty$, and $S \in H_\infty$ (Note that, for $S \in H_\infty$, $\|S\|_{H_\infty} = \|S\|_{L_\infty} = \|S\|_\infty$, where L_∞ is the class of bounded, measurable functions on the imaginary axis). Hence, the peak value is precisely the norm of the system induced by L_2 norms on the input and output signals. This norm is known as the infinity norm of the system.

In this control problem, H_∞ optimization is concerned with

- stabilization of the feedback system, i.e. making all four transfer functions S, CS, SP, and $I + CSP$ stable (in H_∞),
- achieving certain performance specifications by minimizing the H_∞ norm of certain transfer function(s).

As we have seen in the above example, minimizing system norms is a natural way to describe performance specifications. It is useful to be aware of this when studying theoretical papers on H_∞ optimization.

Formulated and solved initially for finite-dimensional systems (see for example Doyle [24], Green et al. [33], Francis [27], Meinsma [49]), the H_∞ control problems were extended to infinite-dimensional systems (see for example Curtain and Green

[15], Barbu [8], van Keulen [71], Foias et al. [25]), Staffans [70]. Engineers and mathematicians used different techniques for solving the H_∞ control problems for infinite-dimensional systems. The mathematical approach can be categorized as

· semigroup approaches,

· partial differential approaches,

· frequency-domain approaches,

· synthesis of state-space and frequency-domain approaches.

In H_∞ control for finite-dimensional systems, the state-space methods are popular because one can solve the problem using Algebraic Riccati Equations (ARE). Although it is possible to extend these results to infinite-dimensional systems, one has to be very careful since semigroup theory and operator valued Riccati equation (ORE) are more complicated. Moreover, the difficulties increase for classes of well-posed systems which do not have a state-space representation in terms of bounded linear operators (see Pritchard and Salamon [57], M. Weiss [77, 78], G. Weiss [75, 76], Staffans [69], Curtain et al. [17], Mikkola [52] e.a.).

Frequency-domain techniques are frequently used to solving H_∞ control problems for infinite-dimensional systems. Over the years there were voices claiming the advantages of the frequency-domain methods. One of the first papers in this direction was published by Horowitz and Shaked [35]. They claim that “extremely important factors of sensor noise and loop bandwidths are obscured by the state-space formulation” and “the important practical considerations are constraints that have been clearly revealed and considered” in the frequency-domain formulation.

In this thesis a frequency-domain approach to H_∞ control problems for infinite-dimensional systems is taken. More precisely, we consider a J-spectral factorization approach to solving H_∞ control problems for multiple-input multiple-output (MIMO) infinite-dimensional systems with their transfer functions in the Wiener class (the corresponding impulse responses are absolutely integrable). Before we clarify what is the Wiener class of transfer functions, we present the notion of J-spectral factorization and how it appears in H_∞ control.

1.1 J-spectral factorization

Given a matrix-valued function Z defined on the imaginary axis, the *J-spectral factorization* problem is to find a stable invertible matrix-valued function V with a stable inverse (we call such matrix-valued functions *bistable*) such that

$$Z(s) = V^*(s) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} V(s).$$

for almost all s on the imaginary axis, by V^* we denote the transposed conjugate of V. The matrix-valued function V is called a J-spectral factor for Z.

To illustrate how the J-spectral factorization appears in H_∞ control we consider as an example the *mixed sensitivity problem.* Consider the closed-loop of Figure 1.2. Let the functions W_1, W_2 and V be given. The sub-optimal mixed sensitivity

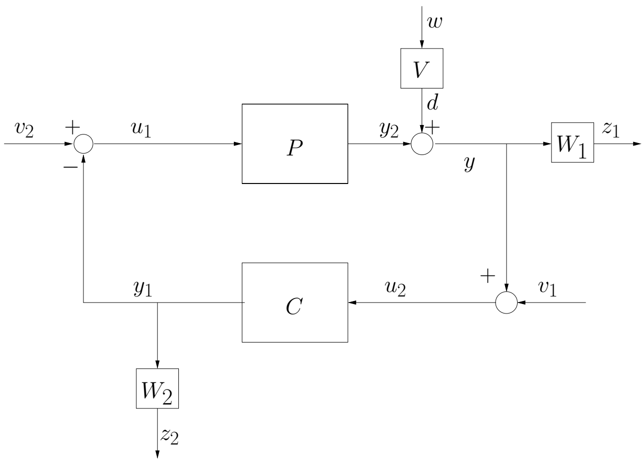

Figure 1.2: Mixed sensitivity problem diagram.

problem is to find a compensator C such that it stabilizes the given plant P and minimizes the infinity norm of the transfer matrix

$$\begin{bmatrix} W_1SV \\ W_2CSV \end{bmatrix}. \tag{1.2}$$

The compensator C and the plant P are assumed to be in $F_\infty := H_\infty/H_\infty$ (the quotient field of H_∞). We say that C stabilizes P if the transfer matrices from the input signals to the output signals are stable (in H_∞).

This is a version (Kwakernaak [47, 48]) of what is known as the mixed sensitivity problem (Verma and Jonckheere [72]). The name comes from the fact that the optimization involves both the sensitivity, S, and the input sensitivity function (or, in other versions, the complementary sensitivity function), CS. By choosing the functions W_1, W_2 and V suitably, S and CS may be made small in appropriate frequency regions. In particular, with W_1V large at low frequencies and W_2V large at high frequencies, the solution of the mixed sensitivity problem has the property that the first term of the criterion dominates at low frequencies and the second at high frequencies. Hence, minimizing the criterion, S will be small at low frequencies and CS small at high frequencies.

In order to reformulate the mixed sensitivity problem into a simpler problem, we need the notion of coprime factorization. Vidyasagar [73] recognized that in many situations it is possible to model an unstable plant as a "ratio" of two stable transfer functions that are coprime (a Bezout identity is satisfied). A result of Smith [68] says that if C stabilizes P, then both, the plant and the compensator, admit coprime factorizations $P = P_d^{-1}P_n$ and $C = C_nC_d^{-1}$, with $P_dC_d + P_nC_n = 1$. Using this result we see that

$$\begin{aligned} S &= (1+PC)^{-1} \\ &= \left[P_d^{-1}(P_dC_d + P_nC_n)C_d^{-1}\right]^{-1} \\ &= C_dP_d, \end{aligned}$$

and

$$CS = C_nC_d^{-1}C_dP_d = C_nP_d.$$

Hence, the mixed sensitivity problem becomes

$$\min_{\substack{C_n, C_d \in H_\infty \\ P_dC_d + P_nC_n = 1}} \left\| \begin{bmatrix} W_1C_dP_dV \\ W_2C_nP_dV \end{bmatrix} \right\|_\infty$$

Here one can see the power of the coprime factorization. The non-linear problem (1.2) has been formulated into a problem which is linear in its unknowns.

Suppose, for simplicity, that P is a stable plant and so we may take $P_d = 1$. Furthermore, assume that $V = 1$. Recall that we have to minimize the infinity norm of the transfer matrix

$$\begin{bmatrix} W_1S \\ W_2CS \end{bmatrix} = \begin{bmatrix} W_1C_d \\ W_2C_n \end{bmatrix}$$

over all stable pairs (C_n, C_d) such that $C_d + P_nC_n = 1$. Let γ be an upper bound for the infinity norm, i.e.,

$$\left\| \begin{bmatrix} W_1C_d \\ W_2C_n \end{bmatrix} \right\|_\infty < \gamma.$$

This holds if and only if there exists an $\varepsilon_1 > 0$ such that

$$\left\| \begin{bmatrix} W_1(s)C_d(s) \\ W_2(s)C_n(s) \end{bmatrix} \right\| < \gamma - \varepsilon_1$$

for all s on $\mathbb{C}_0 \cup \{\infty\}$ (the compactified imaginary axis). Or equivalently,

$$\begin{bmatrix} W_1(s)C_d(s) \\ W_2(s)C_n(s) \end{bmatrix}^* \begin{bmatrix} W_1(s)C_d(s) \\ W_2(s)C_n(s) \end{bmatrix} - \gamma^2 < -\varepsilon_2 < 0$$

for all s on $\mathbb{C}_0 \cup \{\infty\}$. The above inequality can be written as

$$\begin{bmatrix} W_1(s)C_d(s) \\ W_2(s)C_n(s) \\ 1 \end{bmatrix}^* J_\gamma \begin{bmatrix} W_1(s)C_d(s) \\ W_2(s)C_n(s) \\ 1 \end{bmatrix} < 0,$$

where J_γ is

$$J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 \end{bmatrix}.$$

Defining M as

$$M := \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \\ P_n & P_d \end{bmatrix},$$

and using the relations on C_n and C_d, we obtain

$$\begin{bmatrix} W_1 C_d \\ W_2 C_n \\ 1 \end{bmatrix} = M \begin{bmatrix} C_n \\ C_d \end{bmatrix}.$$

So we have now derived the problem: Find stable C_n and C_d such that

$$C_d + P_n C_n = 1$$

and

$$\begin{bmatrix} C_n \\ C_d \end{bmatrix}^* M^* J_\gamma M \begin{bmatrix} C_n \\ C_d \end{bmatrix} < 0.$$

Suppose now that W is a J-spectral factor for $M^* J_\gamma M$. Then we have the equivalence

$$\begin{bmatrix} C_n \\ C_d \end{bmatrix}^* M^* J_\gamma M \begin{bmatrix} C_n \\ C_d \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{bmatrix} C_n \\ C_d \end{bmatrix}^* W^* J_1 W \begin{bmatrix} C_n \\ C_d \end{bmatrix} < 0.$$

This last inequality has a solution. Namely, C_n and C_d given by

$$\begin{bmatrix} C_n \\ C_d \end{bmatrix} = W^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

satisfy

$$\begin{bmatrix} C_n \\ C_d \end{bmatrix}^* W^* J_1 W \begin{bmatrix} C_n \\ C_d \end{bmatrix} = -1 < 0.$$

From the above calculations we see that this approach reduces the mixed-sensitivity problem to a J-spectral factorization problem. The power of J-spectral factorization consists in the fact that it leads to a solution of the mixed sensitivity problem.

1.2 The standard H_∞ control problem

The standard H_∞ control problem was introduced in 1984 by J.C. Doyle [24]. In a few words, the H_∞ sub-optimal control problem is to find a controller which stabilizes a given plant and which makes the infinity norm of the associated transfer function less than a given positive real number. It is called standard because it includes many H_∞ control problems as special cases.

The standard H_∞ control problem is shown in Figure 1.3, where w, u, z and y may be vector valued signals. Tipically we call

- w is the exogenous input

- u is the control signal
- z is the output to be controlled
- y is the measured output

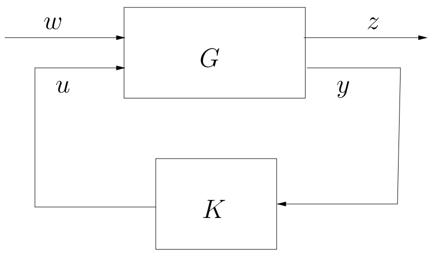

Figure 1.3: The standard H_∞-control problem.

The transfer matrices G and K are assumed to be in $F_\infty := H_\infty/H_\infty$. Let G be a stabilizable plant, and partition G compatibly with the sizes of the inputs and outputs

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{1.3}$$

To define what we mean by "K internally stabilizes G" (often internally will be omitted), we introduce two additional inputs, v_1 and v_2, as in Figure 1.4.

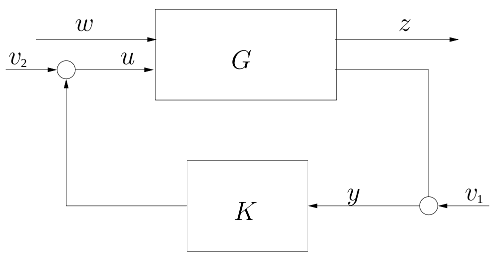

Figure 1.4: Stability diagram.

The algebraic equations corresponding to Figure 1.4 can be represented in the following block format

$$\begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -K & I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}.$$

For well posedness of the closed-loop system, we need to guarantee that the matrix-valued function on the left-hand side is invertible. A sufficient condition for this to hold is that $\det(I - G_{22}(s)K(s)) \neq 0$ for almost all s in the right half-plane. We say that K internally stabilizes G if the nine transfer matrix-valued function from w, v_1, v_2 to z, y, u are stable (in H_∞).

Next we define the sub-optimal H_∞ control problem.

Definition 1.1 (The standard H_∞ sub-optimal control problem) *Given a stabilizable plant $G \in F_\infty$ and a positive bound γ, the standard H_∞ sub-optimal control problem is to find a compensator $K \in F_\infty$ such that K internally stabilizes G, and the transfer function T_{zw}, from w to z, satisfies*

$$\|T_{zw}\|_{H_\infty} < \gamma.$$

If it is possible, describe the general form of all stabilizing controllers which satisfies the above inequality.

The sub-optimal mixed sensitivity problem can be seen as a standard H_∞ sub-optimal control problem. This is defining

$$G := \begin{bmatrix} W_1 & W_1 P \\ 0 & -W_2 \\ I & P \end{bmatrix}$$

and $K := C$ (for the inputs d and y_1, and the outputs z_1, z_2 and y)

We present the *model matching problem* as another example of the standard H_∞ control problem. In Figure 1.5, the transfer matrix T_1 represents a stable model

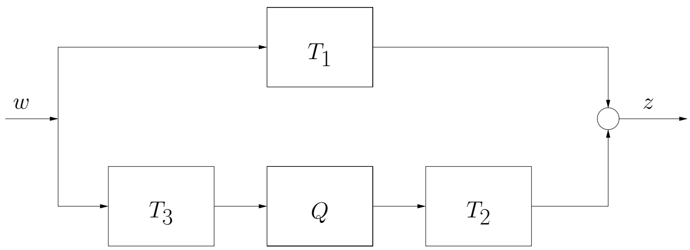

Figure 1.5: Model matching problem diagram.

which is to be mached by the cascade T_2QT_3 of three stable transfer matrix-valued functions. Here $T_i (i := 1..3)$ are given and the "controller" Q is to be designed. The *model matching criterion* is to minimize

$$\|T_1 - T_2QT_3\|_\infty$$

over all stable Q's. The model matching problem can be recast as a standard H_∞ sub-optimal control problem by defining

$$\begin{aligned} G &:= \begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} \\ K &:= -Q, \end{aligned}$$

so that Figure 1.5 becomes equivalent to Figure 1.3.

We will reformulate the model matching problem into another equivalent problem. Consider the model matching problem as defined before, and take, for simplicity,

$$T_2 := \frac{(s-1)(s-3)}{(s+2)^2} \text{ and } T_3 := 1.$$

We can write $T_2 = T_i T_o$, where

$$T_i := \frac{(s-1)(s-3)}{(s+1)(s+3)} \text{ and } T_o := \frac{(s+1)(s+3)}{(s+2)^2}.$$

Denote G and K by

$$G := T_i^{-1} T_1 \text{ and } K := T_o^{-1} Q.$$

Note that, since T_o and T_o^{-1} are stable, K is stable if and only if Q is stable. Since $|T_i(s)| = 1$ for all $s \in \mathbb{C}_0$, it follows that the model matching criterion is the same as the minimization of

$$\|G + K\|_\infty$$

over all stable K's. Note that G has unstable poles, but is bounded on the imaginary axis. This leads us to the *Nehari (extension) problem*: Given $G \in L_\infty$, find

$$\operatorname*{dist}_{L_\infty}(G, H_\infty) := \inf_{K \in H_\infty} \|G + K\|_\infty.$$

This is called the *Nehari problem.* Moreover to find (all) $K_0 \in H_\infty$ such that

$$\operatorname*{dist}_{L_\infty}(G, H_\infty) = \|G + K_0\|_\infty.$$

is referred to as the *Nehari extension problem.*

Before we can formulate the solution to the Nehari problem, we need to introduce the Hankel operator. Let $G \in L_\infty$ be a given function. The Hankel operator with symbol G is denoted by H_G and is defined as

$$H_G : H_2 \to H_2^\perp$$

$$H_G u = \Pi_- G u \text{ for } u \in H_2,$$

where Π_- is the projection operator from L_2 onto $H_2^\perp$. H_2 is the space of functions $u(s)$ analytic in the open right half-plane with the norm

$$\|u\|_2^2 := \sup_{\zeta > 0} \int_{-\infty}^{\infty} \|u(\zeta + j\omega)\|^2 d\omega.$$

H_2 is a closed subspace of $L_2(\mathbb{C}_0)$ (the space of square integrable signals), and $H_2^\perp$ is the orthogonal complement of H_2 with respect to the usual inner product in L_2.

The Nehari theorem (see Nehari [54] for the original formulation, and Adamjan et al. [1]) provides a solution to the Nehari problem: for any $G \in L_\infty$

$$\operatorname*{dist}_{L_\infty}(G, H_\infty) = \inf_{K \in H_\infty} \|G + K\|_\infty = \|H_G\|,$$

where $\|H_G\|$ denotes the norm of the Hankel operator. Most of the research on the solutions of the Nehari extension problem is directed to a sub-optimal version of the problem. The *sub-optimal Nehari extension problem* is: Given $G \in L_\infty$, and $\sigma \geq \|H_G\|$, find all $K \in H_\infty$ such that

$$\|G + K\|_\infty \leq \sigma.$$

Note that when $\sigma = \|H_G\|$ we obtain the Nehari extension problem. As we have seen, the sub-optimal model matching problem is equivalent to the sub-optimal Nehari problem, where G and K are

$$G := T_i^{-1}T_1 \text{ and } K := T_o^{-1}Q.$$

Other famous problems, as Caratheodory-Fejer problem or Nevanlinna-Pick problem, can be reduced to Nehari problems (see for example Partington [56]).

1.3 H_∞ control and model reduction

In engineering areas, distributed parameter systems models are derived, and it can be desirable to replace them by low order models without introducing too much error. A model reduction problem can, in general, be stated as follows: Given a stable model G of a system, find a low order (rational) model G_{ap} such that G and G_{ap} are closed in some sense. The infinity norm is a way to express closeness. In this context, ideal would be to solve the following problem: Given a transfer function $G \in H_\infty$ and a non-negative integer l, find a stable rational function G_{ap} with the MacMillan degree at most l such that the infinity norm

$$\|G + G_{ap}\|_\infty$$

is minimized. To the best of our knowledge, this problem has not been solved yet. Instead of minimizing the infinity norm of the transfer function $G + G_{ap}$, we minimize

$$\|G^\sim + G_{ap}^\sim\|_H.$$

By $\|G^\sim + G_{ap}^\sim\|_H$ we denote the norm of the Hankel operator associated to the (unstable) system $G^\sim + G_{ap}^\sim$, and by $G^\sim$ we denote

$$G^\sim(s) := [G(-\overline{s})]^*,$$

where $\overline{s}$ is the complex conjugate of s. Note that whenever $G \in H_\infty$, $G^\sim$ is bounded and analytic in the open left half-plane. For example

$$\left(\frac{s-1}{s+1}\right)^\sim = \frac{s+1}{s-1},$$

the stable pole -1 is transformed into the unstable pole 1.

So we consider the following problem: Given a transfer function $G \in H_\infty$ and a non-negative integer l, find a stable rational function G_{ap} with the MacMillan degree at most l such that the Hankel norm

$$\|G^\sim + G_{ap}^\sim\|_H \tag{1.8}$$

is small.

In order to formulate the solution, we introduce $H_{\infty,l}$, the space of functions K defined in the right half-plane which are the sum of two functions: one in H_∞ and the other one a rational function K with MacMillan degree at most l, and with all its poles in the open right half-plane.

We formulate the following natural extension of the Nehari problem: Given a transfer function $G \in L_\infty$ and a non-negative integer l, find a function $K \in H_{\infty,l}$ such that the infinity norm

$$\|G^\sim + K\|_\infty \tag{1.9}$$

is small. Suppose that we have a solution K for this problem. Let $K = G_{ap}^\sim + K_{st}$, where $G_{ap}^\sim$ is the (unstable) rational part of K and $K_{st} \in H_\infty$. Then $G_{ap}^\sim$ is a solution to (1.8). In order to illustrate this we need to recall three basic properties of Hankel operators

- $H_{G_1+G_2} = H_{G_1} + H_{G_2}$,
- $H_G = 0$, for $G \in H_\infty$,
- $\|G\|_H \leq \|G\|_\infty$ for any $G \in L_\infty$.

Using these properties, we see that

$$\begin{aligned} H_{G^\sim + K} &= H_{G^\sim + K_{st} + G_{ap}^\sim} = H_{G^\sim + G_{ap}^\sim} + H_{K_{st}} \\ &= H_{G^\sim + G_{ap}^\sim}, \end{aligned}$$

and so

$$\|G^\sim + G_{ap}^\sim\|_H = \|G^\sim + K\|_H \leq \|G^\sim + K\|_\infty.$$

From the above inequality we see that whenever we find a K which makes (1.9) small, the (unstable) rational part of K, $G_{ap}^\sim$, makes (1.8) small.

To say more about rational approximation of infinite-dimensional transfer functions, the compactness of the Hankel operator is required, and we consider this as an assumption in the rest of this section. The following criterion for the compactness of an Hankel operator holds (see Partington [56]).

Theorem 1.2 *Let G be a matrix-valued function in L_∞. The Hankel operator with symbol G is compact if and only if*

$$G \in H_\infty^{n\times m} + C^*,$$

where C^* is the space consisting of those functions continuous on the imaginary axis, with well defined limits at $\pm j\infty$, and they are equal.

Whenever the Hankel operator with symbol G is compact it has countably many singular values (square roots of the eigenvalues of $H_G^* H_G$); we denote the singular values of H_G by σ_k and we count them with respect to their multiplicity. The σ_k's are referred to as the Hankel singular values of G. The norm of H_G equals the largest singular value of H_G.

The minimal value of the Hankel norm (1.8) was found for scalar case by Adamjan et al. [1], and for matrix-valued case by Kung and Lin [46], (for more background on this problem see Sasane [64]). This is stated in the following theorem.

Theorem 1.3 *For any $G^{\sim} \in L_\infty$*

$$\inf_{Q} \|G^{\sim} + Q\|_H = \inf_{K \in H_{\infty,l}} \|G^{\sim} + K\|_\infty = \sigma_{l+1},$$

where $Q^{\sim}$ is a stable rational transfer function with MacMillan degree at most l.

Practically, finding a rational approximant of a non-rational transfer function which achieve the minimum leads us to the *sub-optimal Hankel norm approximation problem*: Given $G \in L_\infty$, and $\sigma_{l+1} < \sigma < \sigma_l$, find all $K \in H_{\infty,l}$ such that

$$\|G + K\|_\infty \leq \sigma.$$

For the case when σ equals a singular value of the Hankel operator we obtain the *optimal Hankel norm approximation problem.*

1.4 The Wiener class of transfer functions

Most of the results in this thesis are presented for the so called Wiener class of transfer functions or for a quotient field of this class. Here we give a brief definition of this class and we refer the reader to the next chapter for more details.

We define the Wiener class of stable transfer functions via impulse responses. Let $f_0 \in \mathbb{C}$ and let $\delta(t)$ be the delta distribution at zero. Consider functions $f(t)$ (in time domain) which are the sum of an integrable function and a delta function, i.e.,

$$f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

with $\int_0^\infty |f_a(t)|dt < \infty$. These functions are Laplace transformable, and their Laplace transform is

$$\hat{f}(s) := f_0 + \int_0^\infty e^{-st} f_a(t)dt.$$

This allows us to define the set $\hat{\mathcal{A}}$ as

$$\hat{\mathcal{A}} := \left\{ \hat{f} \mid f \in \mathcal{A} \right\}.$$

The elements of $\hat{\mathcal{A}}$ are bounded and analytic in the open right half-plane ($\mathrm{Re}(s) > 0$). Thus $\hat{\mathcal{A}} \subset H_\infty$, and we may call the elements of $\hat{\mathcal{A}}$ stable. The set $\hat{\mathcal{A}}$ is known as the *Wiener algebra on the right half-plane* or the *causal Wiener algebra*. We consider also the algebra

$$\hat{\mathcal{W}} = \{f \mid f = f_1 + f_2, \text{ with } f_1, f_2^{\sim} \in \hat{\mathcal{A}}\}. \tag{1.11}$$

We call $\hat{\mathcal{W}}$ the *Wiener class* of transfer functions, or the *Wiener algebra on the imaginary axis*.

In the following two examples we show that these transfer functions appear naturally.

Example 1.4 Consider the ordinary differential equation in x

$$\begin{cases} \frac{d}{dt}x(t) = -x(t) + u(t) \text{ for } t > 0 \\ x(0) = 0, \end{cases}$$

where $u(t)$ is the input variable. As output we choose

$$y(t) = \begin{cases} x(t-1) & , t \geq 1 \\ 0 & , 0 \leq t < 1. \end{cases}$$

The impulse response of the system is $e^{-(t-1)}\mathbf{1}_{[1,\infty)}(t)$, which is an integrable function, and hence its Laplace transform

$$n(s) = \frac{e^{-s}}{s+1}$$

is an element of $\hat{\mathcal{A}}$. ∎

Example 1.5 Consider the integrable impulse response

$$h(t) = \delta(t) - e^{-(t-1)}\mathbf{1}_{[1,\infty)}(t) - (1+a)e^{-t}\mathbf{1}_{[0,\infty)}(t).$$

Its Laplace transform

$$d(s) = \frac{s - a - e^{-s}}{s+1},$$

is an element of $\hat{\mathcal{A}}$. ∎

We define the following quotient field of $\hat{\mathcal{A}}$

$$\hat{\mathcal{B}}_0 := \hat{\mathcal{A}}[\hat{\mathcal{A}}_\infty]^{-1} = \{fg^{-1} | f \in \hat{\mathcal{A}}, g \in \hat{\mathcal{A}}_\infty\},$$

where $\hat{\mathcal{A}}_\infty$ is the subclass of transfer functions in $\hat{\mathcal{A}}$ with the property that their limit at infinity exists it is nonzero, and there are no zeros on the imaginary axis.

Example 1.6 Consider the following retarded equation

$$\begin{cases} \frac{d}{dt}x(t) = ax(t) + x(t-b) + u(t) & ,t \geq 0, \\ x(0) = 0 & \\ x(\alpha) = 0 & ,-b \leq \alpha < 0, \\ y(t) = x(t-1) & ,t \geq 0. \end{cases}$$

where the constants have the following values

$$a = -0.1, \quad b = 2\log\frac{5}{3}.$$

The transfer function $g(s)$ of the system is

$$g(s) = \frac{c_0 e^{-s}}{s + 0.1 - e^{-sb}}.$$

We can write it as

$$g(s) = \frac{\frac{e^{-s}}{s+1}}{\frac{s-a-e^{-sb}}{s+1}} = \frac{n(s)}{d(s)}$$

The only pole in the right half-plane of $g(s)$ is 0.5. It can be checked that $d \in \hat{\mathcal{A}}_\infty$. From Example 1.4 we have that $n \in \hat{\mathcal{A}}$. So $g \in \hat{\mathcal{B}}_0$. Moreover, it can be proved that $(n(s), d(s))$ is a coprime factorization of $g(s)$, i.e., there exist $x, y \in \hat{\mathcal{A}}$ such that $xn + dy = 1$. ■

1.5 Organization of the thesis

In this thesis we develop frequency-domain solutions to H_∞ control problems for linear time invariant (LTI) infinite-dimensional systems for multi input multi output (MIMO) systems. The new idea behind the approach taken in this thesis is to use the factorization results of Clancey and Gohberg [13] in an essential way to obtain a J-spectral factorization of certain matrix-valued functions. With the help of the equalizing vectors (the elements in the kernel of the Toeplitz operator associated to the matrix-valued function to be factorized), we are able to establish the existence of a J-spectral factorization. We do this either directly, for the Nehari problem and the Hankel norm approximation problem, or via the positivity of some signal space for the standard H_∞ sub-optimal control problem.

To facilitate the reading of the thesis, we give a brief description of the material contained in the succeeding chapters.

Chapter 2
The following classes of transfer functions for infinite dimensional systems are recalled

1. Nevanlinna class ($F_\infty := H_\infty / H_\infty$);

2. Wiener class and other algebras of transfer functions;

3. Pritchard-Salamon class (a class of infinite-dimensional systems which admits a state-space representation).

Proofs or references are given for useful results concerning these classes.

Chapter 3
For the Wiener class of matrix-valued functions we provide necessary and sufficient conditions for the existence of a J-spectral factorization. One of these conditions is in terms of equalizing vectors. A second one states that the existence of a J-spectral factorization is equivalent to the invertibility of the Toeplitz operator associated to the function to be factorized. Our proofs are simple and only use standard results of general factorization theory presented in Clancey and Gohberg [13]. Note that we do not use a state-space representation of the system. However, we make the connection with the known results for the Pritchard-Salamon class of systems where an equivalent condition with the solvability of an algebraic Riccati equation is given.

Chapter 4
In this chapter we obtain a simple frequency-domain solution for the sub-optimal Nehari extension problem for the Wiener class of infinite-dimensional systems. The approach is via J-spectral factorization, and it uses the concept of equalizing vectors. The connection between the equalizing vectors and the Nehari extension problem is given. Moreover, we present the sub-optimal Nehari extension problem for Pritchard-Salamon systems as a particular case. Note that, since for this class we have a state-space description, we can write explicitly the J-spectral factor in state-space form.

Chapter 5
The purpose of this chapter is to present an elementary derivation of the reduction of the sub-optimal Hankel norm approximation problem into a J-spectral factorization problem. We do this also for the Wiener class of matrix-valued transfer functions. The Pritchard-Salamon class of systems again fits into this theory and a state-space description for the J-spectral factor can be explicitly written.

Chapter 6
Sufficient conditions for the solvability of the standard H_∞ sub-optimal control problem for the Nevanlinna class are given. The sufficient conditions are formulated in terms of the existence of two J-lossless factorizations. For two classes of infinite-dimensional systems, we prove that the sufficient conditions are also necessary. Furthermore, a formula for the set of all stabilizing controllers is given.

Chapter 7
In this chapter we present some of the results from the previous chapters in the framework of decomposing Banach algebras of continuous functions on the imaginary axis. We present the Wiener algebra as a particular case.

Some hints and directions for ongoing and future research are presented in a concluding chapter.

Chapter 2

Classes of transfer functions

Introduction

Consider a linear, time-invariant system with inputs and outputs and assume that the system is innitially at rest. The relationship between the Laplace transform of the inputs and the outputs can be expressed in terms of a linear map known as the transfer function. More precisely, suppose that we have an input $u : [0, \infty) \to \mathbb{C}^n$ and an output $y : [0, \infty) \to \mathbb{C}^m$ which are Laplace transformable. Let $\hat{u}$ and $\hat{y}$ be their Laplace transforms. A rational or irrational function $G(s)$ which relates $\hat{u}$ and $\hat{y}$ by

$$\hat{y}(s) = G(s)\hat{u}(s) \tag{2.1}$$

in some right half-plane is called the *transfer function.* We do not require $G(s)$ to be defined in the whole complex plane, but analyticity and boundedness in some open right half-plane will normally be assumed.

For systems given in state-space with the usual quadruple $\Sigma(A, B, C, D)$, the transfer function is normally given by

$$G(s) = C(sI - A)^{-1}B + D.$$

Taking the inverse Laplace transform of the transfer function gives the *impulse response* of the system

$$h(t) = \begin{cases} D\delta(t) + CT(t)B & t \geq 0 \\ 0 & t < 0, \end{cases}$$

where $T(t)$ is the strongly continuous semigroup generated by A, and $\delta(t)$ is the delta distribution at zero. Time-domain and frequency-domain properties of linear, time-invariant systems are equally important. Over the years there were voices claiming the advantages of the frequency-domain approach (see for example I.M. Horowitz [35]).

In this chapter we recall some classes of transfer functions. Via (2.1) one can define for every transfer function a linear, time-invariant system. In the first section of this chapter we consider a large class of transfer functions, known as the Nevanlinna class of infinite-dimensional systems.

All absolutely integrable functions on $[0, \infty)$ have a Laplace transform. Moreover, the delta distribution is Laplace transformable. Taking the Laplace transform of a sum of an integrable function and the delta function leads to a class of stable transfer functions (note that these transfer functions are bounded and analytic in the open right half-plane). This will be one of the two classes of stable transfer functions, which we will describe in the second section, and it is known in the literature as the Wiener algebra on the right half-plane. Many of the examples of stable systems which we encounter in applications are included in this algebra.

In order to give a connection with the state-space representation we consider also the Pritchard-Salamon class of systems. This is a class of abstract infinite-dimensional state-space systems allowing some unboundedness of the control and observation operators. We show that the transfer function of a stable Pritchard-Salamon system is strictly included in the Wiener algebra on the right half-plane.

2.1 Nevanlinna class of infinite-dimensional systems

In this section we introduce the largest class of stable transfer functions one can think of when H_∞ control is the main purpose. First we introduce some notation. For any $\varepsilon \in (-\infty, +\infty)$, $\mathbb{C}_\varepsilon$ will denote the vertical line containing ε:

$$\mathbb{C}_\varepsilon := \{s \in \mathbb{C} \mid \operatorname{Re}(s) = \varepsilon\}.$$

By $\mathbb{C}_{\varepsilon,+}$, $\mathbb{C}_{\varepsilon,-}$ we mean the open right and left half-plane, respectively delimited by $\mathbb{C}_\varepsilon$:

$$\begin{aligned} \mathbb{C}_{\varepsilon,+} &:= \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \varepsilon\}, \\ \mathbb{C}_{\varepsilon,-} &:= \{s \in \mathbb{C} \mid \operatorname{Re}(s) < \varepsilon\}. \end{aligned}$$

For $\varepsilon = 0$, $\mathbb{C}_0$ denotes the imaginary axis, and

$$\begin{aligned} \mathbb{C}_+ := \mathbb{C}_{0,+} &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}, \\ \mathbb{C}_- := \mathbb{C}_{0,-} &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}. \end{aligned}$$

We also define

$$\begin{aligned} \overline{\mathbb{C}_+} &:= \mathbb{C}_0 \cup \mathbb{C}_+, \\ \overline{\mathbb{C}_-} &:= \mathbb{C}_0 \cup \mathbb{C}_-. \end{aligned}$$

The function $\phi : \overline{\mathbb{C}_+} \cup \{\infty\} \to \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the unit disc and

$$\phi(s) = \frac{1-s}{1+s}$$

is called the *Cayley transform*. We equip $\mathbb{C}_0 \cup \{\infty\}$ with its one point compactification topology. This is the same as the topology induced by the topology of the unit circle through the Cayley transform. As a consequence we have that $-j\infty$ is identified with $+j\infty$. Similarly, we equip $\overline{\mathbb{C}_+} \cup \{\infty\}$ with the topology induced via Cayley transform by the topology of the closed unit disc.

Definition 2.1 *Let $\overline{\sigma}(A)$ denote the maximum singular value of the matrix A. We consider the Hardy space*

$$\begin{aligned} H_\infty^{n\times m} \quad := \quad \{F : \mathbb{C}_+ \to \mathbb{C}^{n\times m} \mid F \textit{ is analytic in } \mathbb{C}_+ \textit{ and} \\ \|F\|_{H_\infty} = \sup_{s\in\mathbb{C}_+} \overline{\sigma}(F(s)) < \infty\}, \end{aligned}$$

to be the set of our stable matrix-valued transfer functions. So a matrix-valued function F is stable *if it is analytic and bounded in the open right half-plane.*
We say that a square matrix-valued function is bistable *if it is stable, its inverse exists and it is also stable. We denote by $GH_\infty^{n\times n}$ (the units of $H_\infty^{n\times n}$) the set of bistable matrix-valued functions.*

With the quotient field of the Hardy space $H_\infty^{n\times m}$ we define a large class of matrix-valued functions. Since H_∞ is an integral domain its quotient field is well defined (see Definition A.7.17 in Curtain and Zwart [23]).

Definition 2.2 *The quotient field of H_∞ is denoted by F_∞,i.e.,*

$$F_\infty := \{H^{-1}G \mid G \in H_\infty,\ H \in H_\infty,\ H \neq 0\}.$$

We denote by $F_\infty^{n\times m}$ the class of $n \times m$ matrix-valued functions with entries in F_∞. Note that functions in $F_\infty^{n\times m}$ are not necessary analytic in some right half-plane.

In order to prove some properties of the stable matrix-valued functions we introduce the following spaces:

$$\begin{aligned}
&L_2^n := \{f : \mathbb{C}_0 \to \mathbb{C}^n \mid \|f\|_2 := \int_{-\infty}^{+\infty} \|f(j\omega)\|^2 d\omega < \infty\},\\
&H_2^n := \{f : \mathbb{C}_+ \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_+\\
&\qquad\qquad \text{and } \|f\|_{H_2}^2 := \sup_{r>0} \int_{-\infty}^{+\infty} \|f(r+j\omega)\|^2 d\omega < \infty\},\\
&H_2^{n,\perp} := \{f : \mathbb{C}_- \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_-\\
&\qquad\qquad \text{and } \|f\|_{H_2^\perp}^2 := \sup_{r<0} \int_{-\infty}^{+\infty} \|f(r+j\omega)\|^2 d\omega < \infty\},\\
&L_\infty^{n\times m} := \{F : \mathbb{C}_0 \to \mathbb{C}^{n\times m} \mid \|f\|_{L_\infty} := \operatorname*{ess\,sup}_{s\in\mathbb{C}_0} \|f(s)\| < \infty\}.
\end{aligned}$$

In this thesis we regard L_2^n, H_2^n, and $H_2^{n,\perp}$ as signal spaces. The Hardy spaces H_2^n and $H_2^{n,\perp}$ give a direct sum decomposition of the space of square integrable functions L_2^n. We denote by Π_+ and Π_- the projection from L_2^n onto H_2^n, and $H_2^{n,\perp}$, respectively. For more properties of Hardy spaces on the half-plane and the connection with time-domain we refer to Hoffman [34] and Partington [56]. We will usually write H_2, $H_2^\perp$, H_∞, L_∞, or F_∞ when we mean H_2^n, $H_2^{n,\perp}$, $H_\infty^{n\times m}$, $L_\infty^{n\times m}$, or $F_\infty^{n\times m}$ and when there is no danger of confusion.

We introduce the following notation

$$F^\sim(s) = [F(-\overline{s})]^*,$$

where Q^* is the transpose conjugate of the matrix $Q \in \mathbb{C}^{n\times m}$. Note that if $F \in H_\infty$ then $F^\sim$ is bounded and analytic in the open left half-plane. A matrix-valued function G will be called *antistable* if $G^\sim$ is stable.

It is easy to see that

$$F^\sim(s) = F(-\overline{s})^* = F(s)^*,$$

for all $s \in \mathbb{C}_0$.

Example 2.3 We consider the matrix-valued function

$$F(s) = \begin{bmatrix} \frac{s+1}{s-1} & \frac{\frac{s-2}{s+2}}{\frac{e^{-s}-s-3}{s+3}} \\ 0 & \end{bmatrix}.$$

We have that

$$\begin{aligned} F^\sim(s) &= \begin{bmatrix} \frac{s+1}{s-1} & \frac{s-2}{s+2} \\ 0 & \frac{e^{-s}-s-3}{s+3} \end{bmatrix}^\sim = \begin{bmatrix} \frac{-\overline{s}+1}{-\overline{s}-1} & \frac{-\overline{s}-2}{-\overline{s}+2} \\ 0 & \frac{e^{-(-\overline{s})}-(-\overline{s})-3}{-\overline{s}+3} \end{bmatrix}^* \\ &= \begin{bmatrix} \overline{\frac{-\overline{s}+1}{-\overline{s}-1}} & \overline{\frac{-\overline{s}-2}{-\overline{s}+2}} \\ 0 & \overline{\frac{e^{-(-\overline{s})}-(-\overline{s})-3}{-\overline{s}+3}} \end{bmatrix}^T = \begin{bmatrix} \frac{-s+1}{-s-1} & 0 \\ \frac{-s-2}{-s+2} & \frac{e^{s}+s-3}{-s+3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ \frac{s+2}{s-2} & \frac{-e^{s}-s+3}{s-3} \end{bmatrix}. \end{aligned}$$

■

In the following five lemmas we summarize some standard results on stable matrix-valued functions.

Lemma 2.4 *For a matrix-valued function $F \in H_\infty^{n\times m}$, we have that $\|F\|_{H_\infty} < \gamma$ if and only if $F^\sim F - \gamma^2 I < 0$ almost everywhere on the imaginary axis and at infinity.*

Proof: For the matrix-valued function $F \in H_\infty^{n\times m} \subset L_\infty^{n\times m}$ we have the relation

$$\|F\|_{H_\infty} = \|F\|_{L_\infty} = \sup_{u\in L_2^m} \frac{\|Fu\|_2}{\|u\|_2},$$

see [23], Theorem A.6.26. Hence $\|F\|_{H_\infty} < \gamma$ is equivalent to

$$\|Fu\|_2^2 - \gamma^2 \|u\|_2^2 < 0,$$

for all $u \in L_2^m$. Using the definition of $\|\cdot\|_2$ and $A^\sim$, we see that this is equivalent to

$$\int_{-\infty}^{\infty} \left[u^\sim(j\omega) F^\sim(j\omega) F(j\omega) u(j\omega) - \gamma^2 u^\sim(j\omega) u(j\omega)\right] d\omega < 0$$

for all $u \in L_2^m$. Rewriting this gives

$$\int_{-\infty}^{\infty} u^\sim(j\omega) \left[F^\sim(j\omega) F(j\omega) - \gamma^2 I\right] u(j\omega)\, d\omega < 0,$$

for all $u \in L_2^m$. This expression is equivalent with $F^\sim F - \gamma^2 I < 0$ on the imaginary axis. ■

Lemma 2.5 *Consider a matrix valued function $F \in L_\infty^{n\times n}$. The following conditions are equivalent*

1. *F is invertible in $L_\infty^{n\times n}$;*
2. *$F^\sim F > \varepsilon I$ on $\mathbb{C}_0$ for some strictly positive ε;*
3. *$\|Fu\|_{L_2^n} > \sqrt{\varepsilon}\|u\|_{L_2^n}$ for all $u \in L_2^n$, $u \neq 0$.*

Proof: First we will prove the equivalence between the first and the last item.

1.⇒ 3. Suppose that $F \in L_\infty^{n\times n}$ is invertible in $L_\infty^{n\times n}$ and let $L \in L_\infty^{n\times n}$ be its inverse. Then $LFu = u$ for all $u \in L_2^n$ and

$$\|u\|_{L_2^n} = \|LFu\|_{L_2^n} \leq \|L\|_{L_\infty} \|Fu\|_{L_2^n} \Rightarrow \|Fu\|_{L_2^n} \geq \frac{1}{\|L\|_{L_\infty}} \|u\|_{L_2^n}.$$

Choosing $\varepsilon > 0$ such that

$$\frac{1}{\|L\|_{L_\infty}} > \sqrt{\varepsilon},$$

gives the desired result.

3.⇒1. If $\|Fu\|_{L_2^n} > \sqrt{\varepsilon}\,\|u\|_{L_2^n}$ for all nonzero $u \in L_2^n$, then $\ker F = \{0\} \subset L_2^n$, and so the correspondence between the domain and the range is one-to-one and thus F^{-1} exists. Suppose that F^{-1} is not bounded almost everywhere. Then, for any arbitrary $M > 0$, there exists a $v \in L_2$ such that

$$\|F^{-1}v\|_2 \geq M\|v\|_2. \tag{2.2}$$

If we replace now u with $F^{-1}v$ in the inequality

$$\|Fu\|_{L_2^n} > \sqrt{\varepsilon}\,\|u\|_{L_2^n}$$

and we use also (2.2), we obtain that

$$\|v\|_2 = \|FF^{-1}v\|_2 > \sqrt{\varepsilon}\|F^{-1}v\|_2 \geq \sqrt{\varepsilon}M\|v\|_2.$$

The inequality above is not true if we choose M sufficiently large. So we conclude that $F^{-1} \in L_\infty$.

2.$\Leftrightarrow$3. The condition $F^\sim F > \varepsilon I$ on the imaginary axis is equivalent to the following positivity condition

$$\int_{-\infty}^{\infty} u^\sim(j\omega)\left[F^\sim(j\omega)F(j\omega) - \varepsilon I\right]u(j\omega)\,d\omega > 0, \tag{2.3}$$

for all $u \in L_2^m$, $u \neq 0$. Expanding the expression in (2.3) gives

$$\int_{-\infty}^{\infty} [u^\sim(j\omega)F^\sim(j\omega)F(j\omega)u(j\omega) - \varepsilon u^\sim(j\omega)u(j\omega)]d\omega > 0$$

which is equivalent to

$$\|Fu\|_{L_2}^2 \geq \varepsilon\|u\|_{L_2}^2$$

for all $u \in L_2^n$, $u \neq 0$. ■

Using the previous lemma and the fact that F^T is invertible if and only if F is, the following result easily follows.

Lemma 2.6 *A matrix $F \in L_\infty^{n\times n}$ is invertible in $L_\infty^{n\times n}$ if and only if $FF^\sim > \varepsilon I$ for some strictly positive ε.*

For stable matrix-valued transfer functions there exists a similar result.

Lemma 2.7 *A stable matrix-valued transfer function F is bistable if and only if the inequality*

$$F(s)^*F(s) \geq \varepsilon I \tag{2.4}$$

holds for all $s \in \mathbb{C}_+$.

Proof: From (2.4) it follows that for every $s \in \mathbb{C}_+$ the matrix $F(s)$ has an inverse. Standard complex analysis gives that this inverse is an analytic function of $s \in \mathbb{C}_+$. Multiplying (2.4) from the right with this inverse and from the left by the transposed of the inverse gives

$$I \geq \varepsilon F(s)^{-*}F(s)^{-1}.$$

Or equivalently

$$\|F^{-1}(s)u\|^2 \leq \frac{1}{\varepsilon}\|u\|^2$$

for all $s \in \mathbb{C}_+$ and $u \in \mathbb{C}^n$. This shows that $F^{-1} \in H_\infty$. Conversely, let F be a bistable matrix-valued function. Since $F^{-1} \in H_\infty$, there exists a constant M such that

$$\|F^{-1}(s)u\|^2 \leq M\|u\|^2$$

for all $s \in \mathbb{C}_+$ and $u \in \mathbb{C}^n$. This is equivalent to

$$I \geq \frac{1}{M} F(s)^{-*} F(s)^{-1},$$

so we can conclude that (2.4) holds for all $s \in \mathbb{C}_+$. ∎

Lemma 2.8 *Suppose that $F \in L_\infty^{n\times n}$ with $\|F\|_{L_\infty} < 1$. Then F is stable if and only if $(I+F)^{-1}$ is stable.*

Proof: The fact that $(I+F)^{-1}$ is stable implies that F is stable is proved in Lemma 8.3.5 in [23]. Now suppose that F is stable. It has to be shown that $(I+F)^{-1}$ is stable. We prove that $(I+F)^{-1} = \sum_{k=0}^{\infty} (-F)^k$. F is stable implies that the sequence of partial sums

$$f_n = \sum_{k=0}^{n} (-F)^k$$

is also stable. Since F is stable, we have that $\|F\|_{L_\infty} = \|F\|_{H_\infty}$. Furthermore, we see that

$$\|f_n - f_m\|_{H_\infty} \leq \sum_{k=m+1}^{n} \|F\|_{H_\infty}^k, \; m < n.$$

Since $\|F\|_{H_\infty} = \|F\|_{L_\infty} < 1$, this implies that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. H_∞ is a Banach space, so it is complete, and thus the sequence f_n has a limit in H_∞. Let us denote the limit with f. We have that

$$f(I+F) = \lim_{n\to\infty} f_n (I+F) = \lim_{n\to\infty} \sum_{k=0}^{n} (-F)^k (I+F) = \lim_{n\to\infty} \left(I - (-F)^{n+1}\right) = I.$$

The equality

$$f_n(I+F) = (I+F)f_n$$

implies that $(I+F)f = I$. Therefore

$$f = (I+F)^{-1},$$

and since $f \in H_\infty$ we have that $(I+F)^{-1}$ is stable. ∎

A general version of this result will be stated in Chapter 7.

The proof of the next lemma will be omitted, since it is straightforward.

Lemma 2.9 *If the matrix-valued function F is an element of $H_\infty^{n\times m}$ then $F^T \in H_\infty^{m\times n}$ and $\left\|F^T\right\|_{H_\infty} = \|F\|_{H_\infty}$.*

We end this section with a technical result that will be usefull in Chapter 6.

Lemma 2.10 *Consider the stable transfer matrices V_1, V_2, U_1 and U_2, with $V_1 \in H_\infty^{n_z \times n_y}$, $V_2 \in H_\infty^{n_z \times n_z}$, $U_1 \in H_\infty^{n_y \times n_z}$, $U_2 \in H_\infty^{n_z \times n_z}$. Assume that V_2 and U_2 are invertible in $L_\infty^{n_z \times n_z}$, and that the following inequalities are satisfied*

$$\left\|V_2^{-1} V_1\right\|_{L_\infty} \leq 1 \quad \textit{and} \quad \left\|U_1 U_2^{-1}\right\|_{L_\infty} < 1.$$

Then $V_1 U_1 + V_2 U_2$ is bistable if and only if V_2 and U_2 are bistable.

Proof: Since $V_1 \in H_\infty^{n_z \times n_y}$, $V_2 \in H_\infty^{n_z \times n_z}$, $U_1 \in H_\infty^{n_y \times n_z}$, $U_2 \in H_\infty^{n_z \times n_z}$ it follows that $V_1 U_1 + V_2 U_2$ is stable. Therefore it is enough to prove that $(V_1 U_1 + V_2 U_2)^{-1}$ is stable if and only if V_2 and U_2 are bistable.

Suppose that V_2 and U_2 are bistable and define $F = V_2^{-1} V_1 U_1 U_2^{-1}$. Since by assumption V_1, V_2^{-1}, U_1, U_2^{-1} are stable, F is also stable. Furthermore,

$$\|F\|_{L_\infty} = \left\|V_2^{-1} V_1 U_1 U_2^{-1}\right\|_{L_\infty} \leq \left\|V_2^{-1} V_1\right\|_{L_\infty} \left\|U_1 U_2^{-1}\right\|_{L_\infty} < 1.$$

Applying Lemma 2.8 it follows that $(I+F)^{-1}$ is stable. Therefore

$$(V_1 U_1 + V_2 U_2)^{-1} = U_2^{-1} \left(V_2^{-1} V_1 U_1 U_2^{-1} + I\right)^{-1} V_2^{-1} = U_2^{-1} (F + I)^{-1} V_2^{-1}$$

is stable.

Conversely, if $(V_1 U_1 + V_2 U_2)^{-1}$ is stable, then

$$\left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right)^{-1} = U_2 (V_1 U_1 + V_2 U_2)^{-1} V_2$$

is also stable. Since $\left\|V_2^{-1} V_1 U_1 U_2^{-1}\right\|_{L_\infty} < 1$ we can apply Lemma 2.8 and conclude that $V_2^{-1} V_1 U_1 U_2^{-1}$ is stable as well. This trivially implies that $I + V_2^{-1} V_1 U_1 U_2^{-1}$ is stable. Now we see that

$$\begin{aligned} U_2^{-1} &= U_2^{-1} \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right)^{-1} \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right) = \\ &= (V_1 U_1 + V_2 U_2)^{-1} V_2 \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} V_2^{-1} &= \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right) \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right)^{-1} V_2^{-1} \\ &= \left(I + V_2^{-1} V_1 U_1 U_2^{-1}\right) U_2 (V_1 U_1 + V_2 U_2)^{-1}. \end{aligned}$$

So U_2^{-1} and V_2^{-1} are stable. ■

2.2 Algebras of transfer functions

First we define two classes of stable transfer functions via their impulse responses.

Definition 2.11 *Let $\delta(t)$ be the delta distribution at zero.*

1. *We say that* $f \in \mathcal{A}$ *if* f *has the representation*

$$f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{2.5}$$

with $\int_0^\infty |f_a(t)|dt < \infty$*, and* $f_0 \in \mathbb{C}$*.*

2. *We say that* $f \in \mathcal{A}_-$ *if* f *has the representation (2.5) with* $\int_0^\infty e^{\varepsilon t}|f_a(t)|dt < \infty$ *for some* $\varepsilon > 0$*, and* $f_0 \in \mathbb{C}$*.*

Proposition 2.12 $\mathcal{A}$ *is a commutative convolution algebra with identity under the norm*

$$\|f\|_{\mathcal{A}} := \int_0^\infty |f_a(t)|dt + |f_0|. \tag{2.6}$$

The convolution product is defined by

$$(f * h)(t) := \int_0^t f_a(t-s)h_a(s)ds + f_0h_a(t) + h_0f_a(t) + f_0h_0\delta(t),$$

where f *is given by (2.5) and* $h \in \mathcal{A}$ *is defined by*

$$h(t) = \begin{cases} h_a(t) + h_0\delta(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Proof: It can be easily seen that $\mathcal{A}$ is a subalgebra of the algebra considered by Curtain and Zwart in Lemma A.7.46, [23]. ■

The following proposition shows that the Laplace transform is well defined for elements of $\mathcal{A}$.

Proposition 2.13 $f \in \mathcal{A}$ *posseses a Laplace transform* $\hat{f}$ *in* $\overline{\mathbb{C}_+}$ *given by*

$$\hat{f}(s) = \int_0^\infty e^{-st}f_a(t)dt + f_0, \text{ for } s \in \overline{\mathbb{C}_+} \tag{2.9}$$

Proof: The case with $f = f_a$ is covered by Definition A.6.1, Property A.6.2 and Lemma A.6.5 in Curtain and Zwart [23]. Although the delta distribution is not a function, it has the Laplace transform the identity on the whole complex plane.

■

By Proposition 2.13 we see that for every $f \in \mathcal{A}$, $\hat{f}$ is well-defined on $\overline{\mathbb{C}_+}$. This allows us to define the set $\hat{\mathcal{A}}$ as

$$\hat{\mathcal{A}} := \left\{ \hat{f} \mid f \in \mathcal{A} \right\}.$$

The following propositions provide some properties of this set.

Proposition 2.14 *Let* $f \in \mathcal{A}$*, and let* $\hat{f} \in \hat{\mathcal{A}}$ *be the corresponding Laplace transform of* f*. The following properties hold:*

1. $\widehat{f * g} = \hat{f}\hat{g}$ *for* $f, g \in \mathcal{A}$;
2. $\hat{f}$ *is bounded in* $\overline{\mathbb{C}_+}$ *and*
$$\sup_{s \in \overline{\mathbb{C}_+}} |\hat{f}(s)| \leq \|f\|_{\mathcal{A}};$$
3. $\hat{f}$ *is holomorphic on* $\mathbb{C}_+$ *and continuous on* $\overline{\mathbb{C}_+}$;
4. $\hat{f} \in \hat{\mathcal{A}}$ *has the limit* f_0 *at infinity, i.e.*
$$\left\{ |\hat{f}(s) - f_0|; s \in \overline{\mathbb{C}_+}, |s| \geq \rho \right\} \to 0 \quad \text{as } \rho \to \infty;$$
5. $\hat{\mathcal{A}} \subset H_\infty$.

Proof: The results follow similarly as in the proof of Corollary A.7.47 from Curtain and Zwart [23] and from Callier and Desoer [10], [11], [12]. ∎

From the above proposition, we see that $\hat{\mathcal{A}} \subset H_\infty$, and thus we may call the elements of $\hat{\mathcal{A}}$ stable.

Proposition 2.15 $(\hat{\mathcal{A}}, \|\cdot\|_\infty)$ *is a commutative Banach algebra with identity under pointwise addition and multiplication.*

Proof: This follows from the properties of $\mathcal{A}$ and the Laplace transform stated in Proposition 2.12, Proposition 2.13 and Proposition 2.14. ∎

The set $\hat{\mathcal{A}}$ is known in the literature as the *Wiener algebra on the right half-plane* or the *causal Wiener algebra*. In order to allow for unstable transfer functions we extend $\hat{\mathcal{A}}$ with $\hat{\mathcal{A}}^{\sim}$ as introduced in the previous section
$$\hat{\mathcal{A}}^{\sim} = \{\hat{f} \in L_\infty \mid \hat{f}^{\sim} \in \hat{\mathcal{A}}\}.$$

Proposition 2.16 *If* $\hat{f} \in \hat{\mathcal{A}} \cap \hat{\mathcal{A}}^{\sim}$, *then* $\hat{f}$ *is a constant.*

Proof: Since $\hat{f} \in \hat{\mathcal{A}}$, we have that $\hat{f}$ is bounded and continuous in the closed right half-plane and holomorphic in the open right half-plane. From $\hat{f} \in \hat{\mathcal{A}}^{\sim}$ we conclude that $\hat{f}$ is bounded and continuous in the closed left half-plane and holomorphic in the open left-half plane. So we can extend $\hat{f}$ to a bounded holomorphic function in the hole complex plane. Applying now the Liouville's theorem we conclude that $\hat{f}$ is a constant function. ∎

Definition 2.17 *We consider the algebra*
$$\hat{\mathcal{W}} := \hat{\mathcal{A}} \oplus \hat{\mathcal{A}}_0^{\sim}, \tag{2.11}$$
where
$$\hat{\mathcal{A}}_0^{\sim} := \{\hat{f} \in L_\infty \mid \hat{f}^{\sim} \in \hat{\mathcal{A}}, \lim_{|s| \to \infty,\, s \in \mathbb{C}_0} \hat{f}(s) = 0\}. \tag{2.12}$$
We use the notation $\hat{\mathcal{W}}^{n \times m}$ *for the class of* $n \times m$ *matrix-valued functions with entries in* $\hat{\mathcal{W}}$, *and we call it the* Wiener class *of transfer functions.*

We see that $\mathcal{A}_0$ and $\mathcal{W}$ are nothing other but $L_1(0,\infty)$ and $L_1(\mathbb{R}) + \delta(t)\mathbb{C}$, respectively. From Proposition 2.14 we see that the elements of $\hat{\mathcal{W}}$ are bounded, continuous on the imaginary axis and their limit at infinity exists (see also Callier and Desoer [10]). The set $\hat{\mathcal{W}}$ is also known as the *Wiener algebra on the imaginary axis.*

Similarly as $\hat{\mathcal{W}}$ we define

$$\hat{\mathcal{W}}_- = \hat{\mathcal{A}}_- \oplus \hat{\mathcal{A}}^\sim_{-0}. \tag{2.13}$$

where

$$\hat{\mathcal{A}}^\sim_{-0} = \{\hat{f} \in L_\infty \mid \hat{f}^\sim \in \hat{\mathcal{A}}_-, \lim_{|s|\to\infty,\, s\in\mathbb{C}_0} \hat{f}(s) = 0\}.$$

We denote by $\hat{\mathcal{A}}^{n\times m}$, $\hat{\mathcal{A}}^{\sim,n\times m}$, $\hat{\mathcal{A}}_-^{n\times m}$,$\hat{\mathcal{W}}^{n\times m}$, $\hat{\mathcal{W}}_-^{n\times m}$ the classes of $n\times m$ matrix-valued functions with entries in $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}^\sim$, $\hat{\mathcal{A}}_-$, $\hat{\mathcal{W}}$, $\hat{\mathcal{W}}_-$, respectively. As in the previous section, we omit the size of the matrix when there is no danger of confusion.

Proposition 2.18 *Let $Z \in \hat{\mathcal{W}}^{n\times n}(\hat{\mathcal{A}}^{n\times n})$ be such that* $\det(Z)$ *does not vanish at any point on the imaginary axis including infinity (see Proposition 2.14 item 4). Then Z is invertible over $\hat{\mathcal{W}}^{n\times n}(\hat{\mathcal{A}}^{n\times n})$.*

Proof: We first present the proof for $Z \in \hat{\mathcal{W}}$. Since $\det(Z) \in \hat{\mathcal{W}}$ does not vanish at any point on the imaginary axis including infinity, a result of Wiener presented by Ahiezer in [3] says that $(\det(Z))^{-1} \in \hat{\mathcal{W}}$. The inverse of a square matrix Z is given by

$$Z^{-1}(s) = \frac{1}{\det(Z(s))} \operatorname{adj}(Z(s)).$$

The matrix-valued function $\operatorname{adj}(Z)$ is in $\hat{\mathcal{W}}^{n\times n}$, since its components are sums and products of the components of $Z(s)$ which are in $\hat{\mathcal{W}}$ by assumption.

For $\hat{\mathcal{A}}^{n\times n}$ the proof is very similar. The only exeption is that one has to use a rezult of Paley and Wiener [55] to conclude that $(\det(Z))^{-1} \in \hat{\mathcal{A}}$ if $\det(Z)$ does not vanish at any point on the imaginary axis including infinity. ■

The following two propositions provide equivalent conditions for the invertibility of a matrix-valued function over $\hat{\mathcal{W}}^{n\times n}(\hat{\mathcal{A}}^{n\times n})$ (see Callier and Desoer [10]).

Proposition 2.19 *Let $F = F_a + F_0\delta$ be an element of $\mathcal{W}^{n\times n}$. The following statements are equivalent:*

1. *$\hat{F}$ is invertible over $\hat{\mathcal{W}}^{n\times n}$;*
2. $\det F_0 \neq 0$ *and* $\det \hat{F}(s) \neq 0$ *for all $s \in \mathbb{C}_0$;*
3. $\inf\{|\det \hat{F}(s)|; s \in \mathbb{C}_0\} > 0$.

Proposition 2.20 *Let $F = F_a + F_0\delta$ be an element of $\mathcal{A}^{n\times n}$. The following statements are equivalent:*

1. *$\hat{F}$ is invertible over $\hat{\mathcal{A}}^{n\times n}$;*

2. $\det F_0 \neq 0$ *and* $\det \hat{F}(s) \neq 0$ *for all* $s \in \mathbb{C}_+$*;*

3. $\inf\{|\det \hat{F}(s)|; s \in \overline{\mathbb{C}_+}\} > 0.$

Corollary 2.21 *Suppose that* $F \in \hat{\mathcal{W}}^{n\times n}$ *with* $\|F\|_{L_\infty} < 1$*. Then* $F \in \hat{\mathcal{A}}^{n\times n}$ *if and only if* $(I+F)^{-1} \in \hat{\mathcal{A}}^{n\times n}$*.*

Proof: Suppose that $F \in \hat{\mathcal{A}}^{n\times n}$ with $\|F\|_{L_\infty} < 1$. It has to be shown that $(I+F)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. Since $\hat{\mathcal{A}}^{n\times n} \subset H_\infty$ (see Proposition 2.14 item 5.), we have that $F \in H_\infty$. From Lemma 2.8 we conclude that $(I+F)^{-1}$ exists and $(I+F)^{-1} \in H_\infty$, so also $[\det(I+F)]^{-1} \in H_\infty$. This means that $[\det(I+F)]^{-1}$ is bounded in $\mathbb{C}_+$, i.e.,

$$\sup_{s\in\mathbb{C}_+} |[\det(I+F)]^{-1}| = M < \infty.$$

Using Lemma A.6.17 from Curtain and Zwart [23], we see that the supremum does not change if we take it over all $s \in \overline{\mathbb{C}_+}$. For $s \in \overline{\mathbb{C}_+}$ we see that

$$|\det(I+F(s))| = \frac{1}{|[\det(I+F(s))]^{-1}|} \geq \frac{1}{M},$$

and so

$$\inf\{|\det(I+F(s))|;\ s \in \overline{\mathbb{C}_+}\} \geq \frac{1}{M}.$$

Using now Proposition 2.20 we obtain the invertibility of $I+F$ over $\hat{\mathcal{A}}$.

Using once more Lemma 2.8 one can prove similarly the converse implication.

■

Remark 2.22 *For a Banach algebra with identity* e*, we have the following result for the existence of an inverse: if* $\|a\| < 1$*, then the inverse of* $e+a$ *exists (see Curtain and Zwart [23], Lemma A.7.15). Since* $\hat{\mathcal{A}}$ *is a Banach algebra with identity (see Proposition 2.15) one implication of the previous corollary can be also seen as a consequence of this rezult.*

As a consequence of Proposition 2.19 and Lemma 2.5 we have the following result.

Lemma 2.23 *Consider the matrix-valued function* $F \in \hat{\mathcal{W}}^{n\times n}$*. The following statements are equivalent:*

1. F *is invertible over* L_∞*;*

2. F *is invertible over* $\hat{\mathcal{W}}$*;*

3. $F^{\sim}F > \varepsilon I$ *on the imaginary axis for some positive* ε*.*

Similar propositions hold true for $\mathcal{W}_-^{n\times n}$ and $\mathcal{A}_-^{n\times n}$.

We denote by $G\hat{\mathcal{W}}^{n\times n}$, $G\hat{\mathcal{A}}^{n\times n}$, $G\hat{\mathcal{A}}^{\sim,n\times n}$ the set of invertible elements in $\hat{\mathcal{W}}^{n\times n}$, $\hat{\mathcal{A}}^{n\times n}$, $\hat{\mathcal{A}}^{\sim,n\times n}$,

Let us denote by RH_∞ the set of rational proper matrix-valued transfer functions, with their poles the open left half-plane, and RL_∞ the set of rational matrix-valued functions that are proper and have no poles on the imaginary axis.

Definition 2.24 *A Banach algebra of continuous functions on the imaginary axis with a well defined limit at infinity and containing RL_∞ as a dense subset will be called a R-*algebra*.*

Definition 2.25 *Let B be a Banach algebra. We call B a* decomposing Banach algebra *if B has closed subalgebras B_+ and B_-, both containing non-zero elements, such that*

$$B = B_- \oplus B_+,$$

where by $\oplus$ we denote the direct sum.

Proposition 2.26 *The algebra $\hat{\mathcal{W}}$ is a decomposing R-algebra.*

Proof: This property it is easy to obtain from the definition of $\hat{\mathcal{W}}$ and Theorem A.7.56 in Curtain and Zwart [23]. It is also stated in Clancey and Gohberg [13], Chapter 2. ■

Similarly, as F_∞, we can define the quotient field of $\hat{\mathcal{A}}$. However, we restrict this quotient field to a smaller set of fractions.

Definition 2.27 *We define the algebra of fractions*

$$\hat{\mathcal{B}}_0 := \hat{\mathcal{A}}[\hat{\mathcal{A}}_\infty]^{-1} = \{fg^{-1} | f \in \hat{\mathcal{A}}, g \in \hat{\mathcal{A}}_\infty\},$$

where $\hat{\mathcal{A}}_\infty$ is the subclass of transfer functions in $\hat{\mathcal{A}}$ with the property that their limit at infinity is nonzero, and there are no zeros on the imaginary axis. We use the notation $\hat{\mathcal{B}}_0^{n\times m}$ for the class of $n \times m$ matrix-valued functions with entries in $\hat{\mathcal{B}}_0$.

Since a function g in $\hat{\mathcal{A}}_\infty$ has no zeros on the closed imaginary axis (including infinity) the number of unstable zeros of g in $\overline{\mathbb{C}_+}$ is finite (counted with respect to their multiplicity) and all of them are in $\mathbb{C}_+$. Note that the poles of fg^{-1} in the open right-half plane are exactly the zeros of g. This means that fg^{-1} has finitely many unstable poles and all of them are contained in the open right half-plane.

The following lemma provides a sufficient condition for a bounded analytic function to correspond to a Laplace transformable function. This result was proved by Mossaheb in [53].

Lemma 2.28 *Let g be an analytic function on $\mathbb{C}_+$ such that $sg(s)$ is bounded on $\mathbb{C}_+$. Then for any $\varepsilon > 0$ there exists an h such that $g = \hat{h}$ on $\mathbb{C}_{\varepsilon,+}$ and $\int_0^\infty e^{-\varepsilon t}|h(t)|dt < \infty$.*

We provide a simple representation for the algebra of fractions $\hat{\mathcal{B}}_0$.

Lemma 2.29 *For the algebra of fractions $\hat{\mathcal{B}}_0$ we have the following representation: $\hat{h} \in \hat{\mathcal{B}}_0$ if and only if*

$$\hat{h} = \hat{a} + \hat{r},$$

where $\hat{a} \in \hat{\mathcal{A}}$ and $\hat{r}$ is a strictly proper, rational transfer function with all its poles in the open right half-plane.

Proof:
Sufficiency: Since $\hat{a}$ and $\hat{r}$ are in $\hat{\mathcal{B}}_0$ and by assumption $\hat{\mathcal{B}}_0$ is closed under addition, we have that $\hat{h} \in \hat{\mathcal{B}}_0$.

Necessity: We consider $\hat{h} \in \hat{\mathcal{B}}_0$ whose poles in $\mathbb{C}_+$ are p_i of order m_i, $i = 1, ..., l$. Around each pole p_i, $\hat{h}$ has the Laurent expression

$$\hat{h}(s) = \sum_{i=1}^{r} \sum_{j=1}^{m} \frac{a_{i,j}}{(s-p_i)^j} + \hat{a}(s).$$

We take

$$\hat{r}(s) = \sum_{i=1}^{r} \sum_{j=1}^{m} \frac{a_{i,j}}{(s-p_i)^j},$$

which is a strictly proper, rational transfer function with all its poles in the open right half-plane. We see that $\hat{a} \in \hat{\mathcal{B}}_0$, and we will prove that $\hat{a} \in \hat{\mathcal{A}}$. From the definition of $\hat{\mathcal{B}}_0$, there exists $\hat{m} \in \hat{\mathcal{A}}$ and $\hat{n} \in \hat{\mathcal{A}}_\infty$ such that $\hat{a} = \frac{\hat{m}}{\hat{n}}$.

The only unstable zeroes of $\hat{n}$ should be also zeroes for $\hat{m}$ (otherwise we include them in p_i). We prove that $\hat{m}$ and $\hat{n}$ can be choosen to have no common unstable zeroes. Let $p \in \mathbb{C}_+$ be a common zero for $\hat{m}$ and $\hat{n}$. Since $\hat{n} \in \hat{\mathcal{A}}_\infty \subset \hat{\mathcal{A}}$ and $\hat{m} \in \hat{\mathcal{A}}$ we obtain that (see Krein [45])

$$\frac{\hat{n}(s)}{s-p}, \frac{\hat{m}(s)}{s-p} \in \hat{\mathcal{A}}.$$

This implies that $\hat{n}_1$ and $\hat{m}_1$ defined as

$$\begin{aligned} \hat{n}_1(s) &:= \frac{s+1}{s-p}\hat{n}(s) = \hat{n}(s) + (p+1)\frac{\hat{n}(s)}{s-p}, \\ \hat{m}_1(s) &:= \frac{s+1}{s-p}\hat{m}(s) = \hat{m}(s) + (p+1)\frac{\hat{m}(s)}{s-p}, \end{aligned}$$

are also elements of $\hat{\mathcal{A}}$. Moreover, $\hat{a} = \frac{\hat{m}_1}{\hat{n}_1}$, and since $\hat{n} \in \hat{\mathcal{A}}_\infty$ we have that $\hat{n}_1 \in \hat{\mathcal{A}}_\infty$. Using the procedure described above we can eliminate all unstable zeros of $\hat{n}$ without changing the quotient $\hat{a} = \frac{\hat{m}}{\hat{n}}$. Consequently $\hat{m}$ and $\hat{n}$ can be choosen to have no common unstable zeroes. This implies that $\hat{n}$ can be choosen such that does not vanish in the closed right half-plane, including infinity. Using Proposition

2.20 (scalar version; the item 3. is satisfied) we conclude that $\hat{n}$ is invertible over $\hat{\mathcal{A}}$, and hence $\hat{a} = \frac{\hat{m}}{\hat{n}} \in \hat{\mathcal{A}}$. ■

We define another class of unstable irrational systems.

Definition 2.30 *We define the algebra of fractions*

$$\begin{aligned} \hat{\mathcal{B}}_- &:= \hat{\mathcal{A}}_-[\hat{\mathcal{A}}_{\infty,-}]^{-1} \\ &= \left\{ \hat{m}\hat{n}^{-1}; \hat{m} \in \hat{\mathcal{A}}_- \text{ and } \hat{n} \in \hat{\mathcal{A}}_{\infty,-} \right\} \end{aligned}$$

where $\hat{\mathcal{A}}_{\infty,-}$ is the subclass of transfer functions in $\hat{\mathcal{A}}_-$ with a nonzero limit at infinity over the closed right half-plane and only finitely many unstable poles (counted with their multiplicity) in $\mathbb{C}_+$.
We use the notation $\hat{\mathcal{B}}_-^{n\times m}$ for the class of $n \times m$ matrix-valued functions with entries in $\hat{\mathcal{B}}_-$.

Proposition 2.31 *For the algebra of fractions $\hat{\mathcal{B}}_-$ we have the following representation: $\hat{h} \in \hat{\mathcal{B}}_-$ if and only if*

$$\hat{h} = \hat{a} + \hat{r},$$

where $\hat{a} \in \hat{\mathcal{A}}_-$ and $\hat{r}$ is a strictly proper, rational transfer function with all its poles in the closed right half-plane.

Proof: A proof of this result can be found in Curtain and Zwart [23], see Theorem 7.1.16.b. ■

We present important remarks regarding the classes of irrational transfer functions defined before.

Remark 2.32

1. *$\hat{\mathcal{B}}_-$ is smaller than the algebra of fractions defined by Curtain and Zwart in [23], Definition 7.1.6 (Callier-Desoer class for $\beta = 0$). For example, functions of the form*

$$\sum_{n=1}^{\infty} a_n e^{-t_n s},$$

where a_n and t_n are real positive numbers, are not included in $\hat{\mathcal{B}}_-$ but they belong to the Callier-Desoer class.

2. *$\hat{\mathcal{B}}_-$ is the same class as considered by Curtain and Green in [15].*

3. *We have that $\frac{1}{s} \notin \hat{\mathcal{B}}_0$ but $\frac{1}{s} = \frac{\frac{1}{s+1}}{\frac{s}{s+1}} \in \hat{\mathcal{B}}_-$.*

4. *From the previous item of this remark, Propositions 2.29 and 2.31 and the fact that $\hat{\mathcal{A}}_-$ is strictly contained in $\hat{\mathcal{A}}$ we see that neither of $\hat{\mathcal{B}}_-$ nor $\hat{\mathcal{B}}_0$ contains the other.*

2.3 Coprime factorization

The concept of coprime factorization was introduced for transfer matrices by Vidyasagar in [74], and it will play an important role in the chapters 5-6.

Definition 2.33 *A matrix-valued function $G \in F_\infty$ has a* left-coprime factorization *over H_∞ if there exist matrix-valued functions D, N, X, $Y \in H_\infty$, D is square and* $\det(D) \neq 0$*, such that $G = D^{-1}N$ and and the following* Bezout identity

$$DX + NY = I$$

holds.

Definition 2.34 *A matrix-valued function $G \in F_\infty$ has a* right-coprime factorization *over H_∞ if there exist matrix-valued functions D, N, X, $Y \in H_\infty$ such that $G = ND^{-1}$ and and the following* Bezout identity

$$XD + YN = I$$

holds.

The right (left)-coprime factorization is unique except for the possibility of multiplying the "numerator" and "denominator" matrices on the right (left) by a matrix-valued function invertible over H_∞ (see Vidyasagar [74], Chapter 4).
We call a matrix-valued function $G \in F_\infty$ "stabilizable" if there exist a matrix-valued function $K \in F_\infty$ such that the following matrix-valued functions

$$S := (I - GK)^{-1},\ KS,\ SG,\ I + KSG$$

are elements of H_∞. Not every matrix-valued function has a coprime factorization. However, the set of stabilizable elements have such a factorization, as was proved by Smith [68].

Theorem 2.35 *Every stabilizable matrix-valued function $G \in F_\infty$ has a left and right-coprime factorization over H_∞.*

Definition 2.36 *A matrix-valued function $K \in \hat{\mathcal{B}}_-$ ($K \in \hat{\mathcal{B}}_0$) is said to have a* right-coprime factorization *over $\hat{\mathcal{A}}_-$ ($\hat{\mathcal{A}}$) if there exist matrix-valued functions N, M, X and $Y \in \hat{\mathcal{A}}_-$ ($\hat{\mathcal{A}}$), where M is a square matrix-valued function with* $\det(M) \in \hat{\mathcal{A}}_{\infty,-}$ *($\in \hat{\mathcal{A}}_\infty$) such that $K = NM^{-1}$ and the following Bezout identity holds:*

$$XM + YN = I$$

for all $s \in \overline{\mathbb{C}_+}$. Similarly, one can define the left-coprime factorization *over $\hat{\mathcal{A}}_-$ and $\hat{\mathcal{A}}$.*

The following lemma shows that the elements of $\hat{\mathcal{B}}_0^{n\times m}$ have a coprime factorization.

Lemma 2.37 *For any $K \in \hat{\mathcal{B}}_0^{n\times m}$, there exists a right-coprime factorization $K = NM^{-1}$, where M is rational and* $\det M \in \hat{\mathcal{A}}_\infty$ *has a finite number of unstable zeros non of them on the imaginary axis.*

Proof: As a consequence of Lemma 2.29 we have that $K = G + F$ where G is a proper rational matrix-valued transfer function, with all its poles in the open right half-plane, and $F \in \hat{\mathcal{A}}^{n\times m}$.
Let $G = N_1 M^{-1}$ be a right-coprime factorization of G over RH_∞. This always exists and $\det M \in \hat{\mathcal{A}}_\infty$ (see Lemma A.7.37 in Curtain and Zwart [23]). We have that

$$K = F + G = (FM + N_1)M^{-1}. \tag{2.14}$$

We prove that (2.14) is a right-coprime factorization of K over $\hat{\mathcal{A}}$.
Since $G = N_1 M^{-1}$ is a right-coprime factorization of G over RH_∞, there exists $X_1,\ Y_1 \in RH_\infty$ such that

$$X_1 M - Y_1 N_1 = I.$$

We define

$$\begin{aligned} X &= X_1 + Y_1 F \\ Y &= Y_1. \end{aligned}$$

We see that

$$XM - YN = (X_1 + Y_1 F)M - Y_1(FM + N_1) = I.$$

Moreover, from the ring properties of $\hat{\mathcal{A}}$ we have that $X_1 + Y_1 F \in \hat{\mathcal{A}}$ and $M \in \hat{\mathcal{A}}_\infty$.
We can conclude now that (2.14) is a right-coprime factorization of K over $\hat{\mathcal{A}}$. ∎

The coprime factorization of matrix-valued functions in $\hat{\mathcal{B}}_0$ is unique up to multiplication with an inverible element in $\hat{\mathcal{A}}$. This is stated in the following lemma.

Lemma 2.38 *Let $K \in \hat{\mathcal{B}}_0^{n\times m}$ and $K = NM^{-1}$ be a right-coprime factorization of K over $\hat{\mathcal{A}}$. For any matrix-valued function X invertible over $\hat{\mathcal{A}}^{m\times m}$, $K = (NX)(MX)^{-1}$ is another coprime factorization of K over $\hat{\mathcal{A}}$.*
Moreover, any two right-coprime factorizations of $K \in \hat{\mathcal{B}}_0^{n\times m}$ over $\hat{\mathcal{A}}$ are unique up to a multiplication to the right by a matrix-valued function invertible over $\hat{\mathcal{A}}^{m\times m}$.

Proof: The proof is analogous to the parts b. and c. of the proof of Theorem 7.2.8, Curtain and Zwart [23]. ∎

Similar results as in Lemma 2.37 and Lemma 2.38 can be proved for a left-coprime factorization and hold also for $\hat{\mathcal{B}}_-$ (see Curtain and Zwart [23], Theorem 7.2.8).

2.4 Pritchard-Salamon class of infinite-dimensional systems

In this section we introduce a class of infinite-dimensional systems which admits a state-space description allowing some unboundedness of the control and observation operators. This class is well-studied in the literature, see for example Salamon [61], [62], Pritchard and Salamon [58], Curtain et al. [17], and Weiss [78].

First we recall standard concepts from operator theory. Let us denote by $\mathcal{L}(X)$ the set of bounded linear operators on the Hilbert space X.

Definition 2.39 *A* C_0-semigroup *is an operator-valued function* $T(t)$ *from* $[0, +\infty)$ *to* $\mathcal{L}(X)$ *that satisfies the following properties:*

1. $T(t+s) = T(t)T(s)$ *for* $t, s \geq 0$;
2. $T(0) = I$;
3. $\|T(t)z_0 - z_0\| \to 0$ *as* $t \to 0^+$ *for all* $z_0 \in X$.

The growth bound *of the* C_0*-semigroup* $T(t)$*, denoted by* ω_0*, is defined to be*

$$\omega_0 = \inf_{t>0} \frac{1}{t} \log \|T(t)\|.$$

Definition 2.40 *A* C_0*-semigroup,* $T(t)$*, on a Hilbert space* X *is* exponentially stable *if there exist positive constants* M *and* α *such that*

$$\|T(t)\| \leq Me^{-\alpha t} \text{ for } t \geq 0. \tag{2.15}$$

The α *is called the* decay rate *of* $T(t)$.

Before we define the concepts of admissible input and output operators (due to Salamon [60]) we introduce some notation. For two Hilbert spaces X and Y, $X \hookrightarrow Y$ means that $X \subset Y$, X is dense in Y and the canonical injection is continuous. In particular, there exists some constant c such that for all $x \in X$ there holds $\|x\|_Y \leq c\|x\|_X$.

If $T(\cdot)$ is a C_0-semigroup on two Hilbert spaces X and Y, then its infinitesimal generator will be denoted by using the corresponding space as a superscript, e.g. A^X and A^Y. We denote the growth bound on the Hilbert space X by ω^X.

Definition 2.41 *Let* V *and* W *be complex separable Hilbert spaces with* $W \hookrightarrow V$ *and let* $T(\cdot)$ *be a* C_0*-semigroup on* V *which restricts to a* C_0*-semigroup on* W*. An operator* $B \in \mathcal{L}(\mathbb{C}^m, V)$ *is called an* admissible input operator *for* $T(\cdot)$*, with respect to* (W, V)*, if there exists a constant* $\beta > 0$ *and a* $t > 0$ *such that*

$$\int_0^t T(t-s)Bu(s)ds \in W \tag{2.16}$$

and

$$\| \int_0^t T(t-s)Bu(s)ds\|_W \leq \beta \|u\|_{L_2(0,t;\mathbb{C}^m)} \tag{2.17}$$

for all $u(\cdot) \in L_2(0, t; \mathbb{C}^m)$.

Definition 2.42 *Let V and W be complex separable Hilbert spaces with $W \hookrightarrow V$ and let $T(\cdot)$ be a C_0-semigroup on V which restricts to a C_0-semigroup on W. An operator $C \in \mathcal{L}(W, \mathbb{C}^n)$ is called an* admissible output operator *for $T(\cdot)$ with respect to (W, V), if there exists a constant $\gamma > 0$ and a $t > 0$ such that*

$$\|CT(\cdot)x\|_{L_2(0,t;\mathbb{C}^n)} \leq \gamma \|x\|_V \text{ for all } x \in W. \tag{2.18}$$

Remark 2.43 *The definition of an admissible input and output operators is independent of t. This means that if (2.16), (2.17), or (2.18) holds for some $t > 0$, then it can be shown that it holds for all $t > 0$, where β and γ will depend on t in general (see Curtain et al. [17], Remark 2.10, page 11) .*

Now we give the definition of a Pritchard-Salamon system as in Weiss [77]. This class of systems was introduced for the first time in Pritchard and Salamon [58].

Definition 2.44 *Let V and W be complex separable Hilbert spaces with $W \hookrightarrow V$ and let $T(\cdot)$ be a C_0-semigroup on V which restricts to a C_0-semigroup on W. Let $B \in \mathcal{L}(\mathbb{C}^m, V)$ and $C \in \mathcal{L}(W, \mathbb{C}^n)$ be admissible input and output operators for $T(\cdot)$ with respect to (W, V), respectively. Suppose that $D \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$. Under the above assumptions the system given by*

$$\begin{cases} x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \\ \qquad y(t) = Cx(t) + Du(t) \end{cases} \tag{2.19}$$

where $x_0 \in V$, $t \geq 0$ and $u(\cdot) \in L_2(0, t; \mathbb{C}^m)$ is called a Pritchard-Salamon system *and will be denoted by the quadruple $(T(\cdot), B, C, D)$. If, in addition,*

$$D(A^V) \hookrightarrow W, \tag{2.20}$$

the system is called a smooth Pritchard-Salamon. *Moreover, if $T(\cdot)$ is an exponentially stable semigroup on V and W the system is called an* exponentially stable Pritchard Salamon system.

In the following definition we introduce the transfer function of a Pritchard-Salamon system (2.19).

Definition 2.45 *Consider the system (2.19) with $x_0 = 0$ and suppose that $B \in \mathcal{L}(\mathbb{C}^m, V)$ is an admissible input operator for $T(\cdot)$ with respect to (W, V) and $C \in \mathcal{L}(W, \mathbb{C}^n)$. A holomorphic function $G : \mathbb{C}_{\varepsilon,+} \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ is called a* transfer function *of (2.19) if for any admissible input u there holds*

$$\hat{y}(s) = G(s)\hat{u}(s) \text{ for } s \in \mathbb{C}_{\max\{\varepsilon, \lambda(u)\},+}$$

where

$$\lambda(u) := \inf\{\lambda \in \mathbb{R} \mid u(\cdot)e^{-\lambda\cdot} \in L_2(0, \infty; \mathbb{C}^m)\}.$$

Remark 2.46 *From analiticity, it is clear that if* $G_1 : \mathbb{C}_{\varepsilon_1,+} \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ *and* $G_2 : \mathbb{C}_{\varepsilon_2,+} \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ *are two transfer functions of (2.19) then* $G_1(s) = G_2(s)$ *for all* $s \in \mathbb{C}_{\max\{\varepsilon_1,\varepsilon_2\},+}$.

Any Pritchard-Salamon system of the form (2.19) has a well-defined transfer function provided that $B \in \mathcal{L}(\mathbb{C}^m, V)$ is an admissible input operator for $T(\cdot)$ with respect to (W, V) and $C \in \mathcal{L}(W, \mathbb{C}^n)$ (see Curtain et al. [17]).

Proposition 2.47 *Consider the system (2.19) with* $x_0 = 0$ *and* $D = 0$*. Suppose that* $B \in \mathcal{L}(\mathbb{C}^m, V)$ *is an admissible input operator for* $T(\cdot)$ *with respect to* (W, V) *and* $C \in \mathcal{L}(W, \mathbb{C}^n)$*. Let* u *be an admissible input and let* ε *be any number which satisfies* $\varepsilon > \max\{\omega_W, \omega_V, \lambda(u)\}$*. Then the following statements hold true:*

1. $y(\cdot)e^{-\varepsilon\cdot} \in L_1(0,\infty;\mathbb{C}^n) \cap L_2(0,\infty;\mathbb{C}^n)$;
2. $\hat{y}(s) = C(sI - A^V)^{-1}B\hat{u}(s) + D\hat{u}(s)$ *for all* $s \in \mathbb{C}_{\varepsilon,+}$;
3. $C(\cdot I - A^V)^{-1}B \in H_\infty(\mathbb{C}_{\varepsilon,+}, \mathbb{C}^{n\times m})$ *for all* $\varepsilon > \max\{\omega_V, \omega_W\}$,

where

$$H_\infty(\mathbb{C}_{\varepsilon,+}, \mathbb{C}^{n\times m}) := \{G : \mathbb{C}_{\varepsilon,+} \to \mathbb{C}^{n\times m} \mid G \text{ analytic}, \sup_{s\in\mathbb{C}_{\varepsilon,+}} \|G(s)\|_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^n)} < \infty\}$$

It follows in particular that $C(sI - A^V)^{-1}B$ *is a transfer function of (2.19), where* $D = 0$.

Proof: See Curtain et al. [17], Proposition 2.15, page 451. ■

We remark that the growth bounds ω^V and ω^W are not the same, in general (see Curtain et al. [17]).

Lemma 2.48 *The transfer function of an exponentially stable, Pritchard-Salamon system with finite-dimensional input and output spaces is in the class* $\hat{\mathcal{A}}$.

Proof: See Proposition 3.5 (ii) in Curtain et al. [17] (see also Curtain [14]). ■

Chapter 3

J-spectral factorization

Introduction

Given a matrix-valued function Z defined on the imaginary axis, the J-spectral factorization problem is to find a stable invertible matrix-valued function V with a stable inverse such that

$$Z(s) = V^*(s) \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} V(s).$$

for almost all s on the imaginary axis (recall that by V^* we denoted the transposed conjugate of V). The matrix-valued function V is called a J-spectral factor for Z.

Since the J-spectral factorization plays an essential role in H_∞-control, it is important to know if the matrix-valued function Z possesses such a factorization. In this chapter we provide necessary and sufficient conditions for the existence of a J-spectral factorization for a large class of matrix-valued functions.

The first necessary and sufficient condition is given using the notion of the equalizing vectors. For finite-dimensional systems this result was obtained in Meinsma [50], using state-space techniques. The proof presented here is completely in the frequency-domain and uses the factorization results of Clancey and Gohberg [13] in an essential way.

If the matrix-valued function Z is an element of $\hat{\mathcal{W}}$, we prove that the existence of a J-spectral factorization is equivalent to the invertibility of an associated Toeplitz operator. This generalizes a result obtained in Weiss [77], where this was shown to hold for an exponentially stable Pritchard-Salamon system. The proof in [77] is very long and uses Riccati equations.

In the second section of this chapter we provide the connections between the state-space representation of Pritchard-Salamon systems and the frequency domain results obtained for the Wiener class of infinite-dimensional systems.

Assuming that a J-spectral factorization exists for the matrix-valued function $Z \in \hat{\mathcal{W}}$, we provide an algorithm for computing the J-spectral factor. The algorithm is practical if one can solve two algebraic equations involving projection operators.

3.1 J-spectral factorization for the decomposing Banach algebra $\hat{\mathcal{W}}$

Before we give a precise definition of the J-spectral factorization we recall the definition of a standard factorization relative to the imaginary axis.

Definition 3.1 *The matrix-valued function $Z \in \hat{\mathcal{W}}^{k\times k}$ is said to admit a (right-)* standard factorization *relative to the imaginary axis if Z can be decomposed as*

$$Z = Z_- D Z_+, \tag{3.1}$$

with $Z_- \in G\hat{\mathcal{A}}^{k\times k,\sim}$, $Z_+ \in G\hat{\mathcal{A}}^{k\times k}$ and D a diagonal matrix-valued function of the form

$$D(s) = \operatorname{diag}\left[\left(\frac{s-s_{-,1}}{s-s_{+,1}}\right)^{k_1}, ..., \left(\frac{s-s_{-,n}}{s-s_{+,n}}\right)^{k_n}\right], \; s \in \mathbb{C}_0, \tag{3.2}$$

with $s_{-,i} \in \mathbb{C}_-$, $s_{+,i} \in \mathbb{C}_+$, $k_i \in \mathbb{Z}$ and $k_1 \geq ... \geq k_n$. The integers k_i are called (the right-) partial indices *of the factorization. In the case $k_1 = ... = k_n = 0$, so that,*

$$Z = Z_- Z_+, \tag{3.3}$$

then Z is said to admit a (right-) canonical factorization *relative to the imaginary axis.*

The existence of a standard factorization for the decomposing Banach algebra $\hat{\mathcal{W}}$ of matrix-valued continuous functions has been shown in Clancey and Gohberg [13], Chapter II.

Theorem 3.2 *Let $Z \in \hat{\mathcal{W}}^{k\times k}$ be a matrix-valued function. We have that Z satisfies*

$$\det Z(s) \neq 0, \text{ for all } s \in \mathbb{C}_0 \cup \{\infty\}$$

if and only if Z admits a standard factorization relative to the imaginary axis.

Hence every invertible element of $\hat{\mathcal{W}}$ admits a standard factorization. As mentioned in the introduction we are interested in J-spectral factorizations. For its definition we need to consider the matrix

$$J_{\gamma,n,m} := \begin{bmatrix} I_n & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix},$$

where γ is a strictly positive real number and $n, m \in \mathbb{N}$. Sometimes we simply use $J_{n,m}$, when $\gamma = 1$, or J without indices, when there is no danger for confusion.

Definition 3.3 *Let $Z = Z^{\sim} \in \hat{\mathcal{W}}^{k\times k}$ be a matrix-valued function. Z has a* J-spectral factorization *if there exists a matrix-valued function $V \in G\hat{\mathcal{A}}^{k\times k}$ such that*

$$Z(s) = V^{\sim}(s)JV(s) \text{ for all } s \in \mathbb{C}_0 \cup \{\infty\}. \tag{3.4}$$

A solution V of (3.4) is called J-spectral factor *of the matrix-valued function Z.*

The following theorem provides the relation between different solutions for the J-spectral factorization problem for a matrix-valued function $Z \in \hat{\mathcal{W}}$, in case that the problem is solvable.

Theorem 3.4 *Let $Z = Z^{\sim} \in \hat{\mathcal{W}}^{k\times k}$, and suppose that $X \in G\hat{\mathcal{A}}^{k\times k}$ is a J-spectral factor for Z. Then an $Y \in G\hat{\mathcal{A}}^{k\times k}$ satisfies*

$$Y^{\sim}JY = Z = X^{\sim}JX \tag{3.5}$$

on the imaginary axis if and only if $Y = QX$, where Q is a constant matrix satisfying

$$Q^{\sim}JQ = J. \tag{3.6}$$

Proof: Suppose that there exist bistable matrix-valued functions X and Y such that (3.5) holds on the imaginary axis. Multiplying the equality (3.5) to the right with the inverse of X and to the left with the inverse of $Y^{\sim}$ we obtain that

$$JYX^{-1} = (Y^{\sim})^{-1}X^{\sim}J$$

over the imaginary axis. We define

$$Q := JYX^{-1}.$$

The left-hand side of the above equality is stable and the right-hand side is anti-stable. Using Proposition 2.16, we conclude that Q is a constant matrix. We see that Q satisfies

$$Q^{\sim}JQ = X^{-\sim}Y^{\sim}J^{\sim}JJYX^{-1} = X^{-\sim}(Y^{\sim}JY)X^{-1} = J,$$

which is exactly (3.6).

Conversely, if X is a J-spectral factor for the matrix-valued function Z and Q is a constant matrix such that (3.6) is satisfied, then $QX \in G\hat{\mathcal{A}}$ and

$$(QX)^{\sim}J(QX) = X^{\sim}Q^{\sim}JQX = X^{\sim}JX = J.$$

This implies that QX is also a J-spectral factor for Z. ■

In fact the previous lemma shows that the J-spectral factor is unique up to a multiplication with a constant matrix satisfying the relation (3.6).

One of the necessary and sufficient conditions for the existence of the J-spectral factorization will be given in terms of Toeplitz operators, defined as follows.

Definition 3.5 *Let $G \in L_\infty^{n\times m}$. The* Toeplitz operator *with symbol G is defined by*

$$T_G : H_2^m \to H_2^n,\ T_G x = P_+ G x, \tag{3.7}$$

where $P_+ : L_2^n \to H_2^n$ is the orthogonal projection from L_2^n onto H_2^n.

A property of Toeplitz operators which we will use later is given in the following lemma.

Lemma 3.6 *Let us consider the matrix-valued functions $G, K \in L_\infty$. If $G^\sim \in H_\infty$ or $K \in H_\infty$, then the following equality holds*

$$T_{GK} = T_G T_K. \tag{3.8}$$

Proof: A proof can be found in Weiss [77]. ■

Another necessary and sufficient condition for the existence of the J-spectral factorization will be stated using the equalizing vectors. For the moment we give only the definition of the equalizing vectors. More properties will be provided in the next chapter.

Definition 3.7 *A vector u is an* equalizing vector *of $Z \in \hat{\mathcal{W}}^{k_1\times k_2}$ if u is a nonzero element of $H_2^{k_2}$ with Zu in $H_2^{k_1,\perp}$.*

The following lemma shows that the nonexistence of equalizing vectors is a necessary condition for the existence of a J-spectral factorization (see also Meinsma [50]).

Lemma 3.8 *Let $Z \in \hat{\mathcal{W}} \in G\hat{\mathcal{A}}^{k\times k}$ be a matrix-valued function that admits a J-spectral factorization. Then Z has no equalizing vectors.*

Proof: Suppose that Z admits a J-spectral factorization

$$Z(s) = V^\sim(s) J V(s) \tag{3.9}$$

with $V \in G\hat{\mathcal{A}}$, and let u be an equalizing vector for the matrix-valued function Z. Applying (3.9) to u and multiplying it to the left with the inverse of $V^\sim$ (let us denote it by $V^{-\sim}$), we obtain

$$V^{-\sim} Z u = J V u. \tag{3.10}$$

Since $V \in G\hat{\mathcal{A}}$ and $u \in H_2$, the right-hand side of the equality (3.10) is in H_2. From the definition of an equalizing vector we have that $Zu \in H_2^\perp$. Furthermore, since $V \in G\hat{\mathcal{A}}^+$ and so $V^{-\sim} \in G\hat{\mathcal{A}}^\sim$, we have that the left-hand side of the equality (3.10) is in $H_2^\perp$. Thus equality (3.10) is satisfied only when $u = 0$. But, by definition, the equalizing vectors are nonzero. This means that the matrix-valued function Z has no equalizing vectors. ■

The main result of this chapter is the following.

Theorem 3.9 *Let $Z = Z^{\sim} \in \hat{\mathcal{W}}^{k\times k}$ be such that* $\det Z(s) \neq 0$*, for all $s \in \mathbb{C}_0 \cup \{\infty\}$. The following statements are equivalent:*

1. *Z admits a J-spectral factorization;*
2. *Z has no equalizing vectors;*
3. *The Toeplitz operator T_Z is boundedly invertible.*

Proof: 1. $\Rightarrow$ 3. Suppose that the matrix-valued function Z admits a J-spectral factorization $Z = V^{\sim}JV$, where $V \in G\hat{\mathcal{A}}$. We define $Z_- = V^{\sim}J$ and $Z_+ = V$. Obviously, $Z_+ \in G\hat{\mathcal{A}}$ and $Z_- \in G\hat{\mathcal{A}}^{\sim}$. Since $\hat{\mathcal{A}} \subset H_\infty$, we obtain by Lemma 3.6 that

$$T_{Z_+^{-1}}T_{Z_+} = T_{Z_+^{-1}Z_+} = I = T_{Z_+}T_{Z_+^{-1}}.$$

Thus T_{Z_+} is invertible and $T_{Z_+}^{-1} = T_{Z_+^{-1}}$. Similarly, one can show that $T_{Z_-}^{-1} = T_{Z_-^{-1}}$. From this it is easy to see that $T_{Z_+}^{-1}T_{Z_-}^{-1}$ is the bounded inverse of the Toeplitz operator T_Z (see also Weiss [77], page 25).

3. $\Rightarrow$ 2. Suppose that u is an equalizing vector for the matrix-valued function Z. Since $Zu \in H_2^{\perp}$ we have that

$$T_Z u = P_+ Z u = 0$$

which means that the Toeplitz operator T_Z is not injective, and so T_Z is not invertible.

2. $\Rightarrow$ 1. Since $\det Z(s) \neq 0$ for all $s \in \mathbb{C}_0 \cup \{\infty\}$, we have from Theorem 3.2 that the matrix-valued function Z admits a factorization relative to the imaginary axis in $\hat{\mathcal{W}}$. Let this factorization be given as

$$Z = Z_- D Z_+, \tag{3.11}$$

where $Z_+ \in G\hat{\mathcal{A}}$, $Z_- \in G\hat{\mathcal{A}}^{\sim}$, and D a diagonal matrix function of the form (3.2) (see Definition 3.1). It remains to prove that this standard factorization leeds to a J-spectral factorization. First we show that this factorization is canonical. Suppose that the factorization (3.11) is not canonical, then there exists a $k_i \neq 0$. Since $k_1 \geq k_i$, we have that $k_1 \neq 0$. Let $u \in H_2$ be such that

$$Z_+ u = \begin{bmatrix} \left(\frac{1}{s - s_{-,1}}\right)^{k_1} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

where $s_{-,1} \in \mathbb{C}_-$. This is possible because $Z_+ \in G\hat{\mathcal{A}} \subset H_\infty$, and $(\frac{1}{s-s_{-,1}})^{k_1} \in H_2$. We have that

$$DZ_+u = \operatorname{diag}\left[\left(\frac{s-s_{-,1}}{s-s_{+,1}}\right)^{k_1}, ..., \left(\frac{s-s_{-,n}}{s-s_{+,n}}\right)^{k_n}\right] \begin{bmatrix} (\frac{1}{s-s_{-,1}})^{k_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{1}{s-s_{+,1}})^{k_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix},$$

where $s_{+,1} \in \mathbb{C}_+$, which means that $DZ_+u \in H_2^\perp$. Since $Z_-^\sim \in G\hat{\mathcal{A}} \subset H_\infty$, we have that $Z_-DZ_+u \in H_2^\perp$. Since $Z = Z_-DZ_+$, this shows that u is an equalizing vector for Z. This is in contradiction with our assumption, and thus we conclude that the factorization (3.11) is canonical. Hence we have

$$Z_-Z_+ = Z = Z^\sim = Z_+^\sim Z_-^\sim. \tag{3.12}$$

We rewrite (3.12) as

$$(Z_+^\sim)^{-1}Z_- = Z_-^\sim Z_+^{-1}. \tag{3.13}$$

The left-hand side is an element of $\hat{\mathcal{A}}^\sim$ and the right-hand side is an element of $\hat{\mathcal{A}}$. Thus, from Proposition 2.16, it follows that $Z_-^\sim Z_+^{-1}$ is a constant matrix, which we denote by C. The equation (3.13) shows that $C = C^*$. Since $C = Z_-^\sim Z_+^{-1}$, $Z_-^\sim$ and Z_+^{-1} are invertible, we have that $\det(C) \neq 0$. Thus there exists an unitary matrix U such that

$$C = U^\sim JU.$$

Using once again (3.13) we get that $Z_- = Z_+^\sim U^\sim JU$, and thus

$$Z = Z_+^\sim U^\sim JUZ_+.$$

From the above equality we see that $V = UZ_+$ is a J-spectral factor for the matrix-valued function Z. ■

Remark 3.10

1. *As a consequence of Theorem 3.9 we have the following equivalence: A matrix-valued function $Z \in \hat{\mathcal{W}}$ admits a J-spectral factorization if and only if Z admits a canonical factorization and $Z = Z^\sim$.*

2. *The equalizing vectors of a matrix-valued function $Z \in \hat{\mathcal{W}}$ are elements in the kernel of the Toeplitz operator with symbol Z.*

Remark 3.11 *We make some remarks in order to show the connection between the result of Theorem 3.9 and the results presented in Gohberg et al. [30].*

1. *Let $G \in \hat{\mathcal{W}}^{n\times m}$. What we have defined as the Toeplitz operator with symbol G, is in Gohberg et al. [30] defined as the Wiener-Hopf operator with symbol G.*

2. *A linear operator $T : X \to Y$, acting between the complex Banach spaces X and Y, is called a* Fredholm operator *if its range* $\mathrm{Im}T$ *is closed, and the numbers*
$$n(T) := \dim \mathrm{Ker}\, T \text{ and } d(T) := \dim(Y/\mathrm{Im}T)$$
are finite. In this case $\mathrm{ind}(T) := n(T) - d(T)$ *is called the* index *of the operator T. The condition* $\det Z(s) \neq 0$ *for all $s \in \mathbb{C}_0 \cup \{\infty\}$ is equivalent to the fact that the Toeplitz operator with symbol Z is a Fredholm operator (see Gohberg et al. [30], Theorem XXX.10.2).*

3. *The equivalence between the existence of a canonical factorization ($k_1 = ... = k_n = 0$) and item 3. can be seen as a consequence of Theorem XXX.10.1, Gohberg et al. [30].*

4. *Since $Z = Z^{\sim}$, we have that T_Z, the Toeplitz operator with symbol Z, is selfadjoint and* $\dim(\mathrm{Ker}\, T_Z) = \dim(Y/\mathrm{Im}T_Z)$. *Moreover,* $\dim(\mathrm{Ker}\, T_Z)$ *can be described in terms of partial indices of the factorization (see Gohberg et al. [30], Theorem XXX.10.2).*

The above characterizations are completely in frequency domain. In the following section we add time-domain characterizations. We prove that the existence of a J-spectral factorization is equivalent with the existence of a stabilizing solution of an algebraic Riccati equation. We show this for exponentially stable Pritchard-Salamon systems.

3.2 J-spectral factorization for Pritchard-Salamon systems

Before we define the Popov function we need to introduce the concepts of admissible weighting operator and Popov triple.

Definition 3.12 *Let V and W be complex separable Hilbert spaces with $W \hookrightarrow V$ and let $T(\cdot)$ be a C_0-semigroup on V which restricts to a C_0-semigroup on W. An operator $Q = Q^* \in \mathcal{L}(W)$ is said to be an admissible weighting operator for $T(\cdot)$ with respect to (W, V), if for some $t > 0$, there exists an $M > 0$ such that for every $x, y \in W$*
$$\int_0^t |\langle QT(t)x, T(t)y\rangle_W| dt \leq M\|x\|_V\|y\|_V \tag{3.15}$$
and Q is an admissible output operator for $T(\cdot)$ with respect to (W, V).

Notice that the definition of an admissible weighting operator is independent of t (see Curtain et al. [17]).

We give the definition for the concept of Popov triple associated to a Pritchard-Salamon system as in Weiss [77] (see Definition 4.1 page 65).

Definition 3.13 *Let* V, W *be complex separable Hilbert spaces. The* PS(Pritchard-Salamon)-Popov triple *on* $(W \hookrightarrow V, \mathbb{C}^m)$ *is defined to be a triple of the form*

$$\Sigma = (T(\cdot), B, M = \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix}) \tag{3.16}$$

satisfying the assumptions:

1. $T(\cdot)$ *is a* C_0*-semigroup on* V *and* $T(\cdot)$ *restricts to a* C_0*-semigroup on* W;
2. $B \in \mathcal{L}(\mathbb{C}^m, V)$ *is an admissible input operator for* $T(\cdot)$ *with respect to* (W, V);
3. $R = R^* \in \mathbb{C}^{m\times m}$;
4. $N \in \mathcal{L}(W, \mathbb{C}^m)$ *is an admissible output operator for* $T(\cdot)$ *with respect to* (W, V);
5. $Q = Q^* \in \mathcal{L}(W)$ *is an admissible weighting operator for* $T(\cdot)$ *with respect to* (W, V).

A PS-Popov triple will be called smooth *if condition (2.20) is satisfied. A PS-Popov triple will be called* regular *if the operator* R *is boundedly invertible.*

We introduce the Popov function associated to a PS-Popov triple (see also Weiss [78]).

Definition 3.14 *Let* $\Sigma = (T(\cdot), B, M)$ *be a PS-Popov triple. The* Popov function *associated with* Σ *is a function associating to every* $s \in \mathbb{C}_0 \cap \rho(A^V)$ *the following operator*

$$\begin{aligned} \Pi_\Sigma(s) := R + N(s - A^V)^{-1}B + (N(s - A^V)^{-1}B)^* \\ + \overline{(s - A^V)^{-1}B}^{*W} Q(s - A^V)^{-1}B, \end{aligned} \tag{3.17}$$

where $\overline{(s - A^V)^{-1}B}^{*W}$ *denotes the adjoint of* $(s-A^V)^{-1}B$ *not taken as an operator in* $\mathcal{L}(\mathbb{C}^m, V)$, *but in* $\mathcal{L}(\mathbb{C}^m, W)$.

A simple property of the Popov function is that, if $T(\cdot)$ is an exponentially stable semigroup on V and W, then the relation (3.17) holds for every real ω and defines a function in $L_\infty(\mathbb{C}^{m\times m})$. Indeed, in this case $(j\omega - A^V)^{-1}B \in L_\infty(\mathcal{L}(\mathbb{C}^m, W))$ (see Lemma 2.12 in Curtain et al. [17] or Remark 4.7.d in Weiss [77]).

It is easy to verify that, for $R = D^*JD$, we have

$$G^\sim(s)J_{k_+,k_-}G(s) = \Pi_\Sigma(s), \text{ for all } s \in \mathbb{C}_0, \tag{3.18}$$

where $k_+ + k_- = n$, and Σ is the Popov triple and G is the transfer function associated to the Pritchard-Salamon system (3.16). We recall that the transfer function of an exponentially stable Pritchard-Salamon system is in the class $\hat{\mathcal{A}}$ (see Lemma 2.48).

For the Pritchard-Salamon class an equivalent condition for the existence of the J-spectral factorization in terms of Riccati equations is added to our three conditions. We give first the definitions of the Riccati equations and their stabilizing solutions.

Definition 3.15 *Let $\Sigma = (T(\cdot), B, M)$ be a smooth regular PS-Popov triple. The* Riccati equation *associated with Σ is the following equation in the unknown $X = X^* \in \mathcal{L}(V)$*

$$\begin{aligned}&\langle (A^V - BR^{-1}N)x, Xy\rangle_V + \langle Xx, (A^V - BR^{-1}Ny)\rangle_V \\ &-\langle XBR^{-1}B^*Xx, y\rangle_V + \langle (Q - N^*R^{-1}N)x, y\rangle_W = 0\end{aligned} \tag{3.19}$$

for $x, y \in D(A^V)$.

Definition 3.16 *If there exists a self-adjoint $X \in \mathcal{L}(V)$ satisfying the Riccati equation (3.19) such that $T_{BF^c}(\cdot)$ is an exponentially stable semigroup on V which restricts to an exponentially stable semigroup on W, where*

$$F^c = -R^{-1}(B^*X + N),$$

then X is called a stabilizing solution *of the Riccati equation associated with Σ. The admissible feedback F^c will be called a* stabilizing Riccati feedback.

We have the following theorem giving equivalent conditions for the existence of a J-spectral factorization for an exponentially stable Pritchard-Salamon system.

Theorem 3.17 *We consider an exponentially stable Pritchard-Salamon system with finite-dimensional input and output spaces, and let Π_Σ be its associated Popov function. Suppose that $\det \Pi_\Sigma(s) \neq 0$ for all $s \in \mathbb{C}_0 \cup \{\infty\}$. Then the following statements are equivalent*

1. *The Popov function Π_Σ has a J_{m-k_-,k_-}-spectral factorization;*
2. *The Popov function Π_Σ has no equalizing vectors;*
3. *The Toeplitz operator T_{Π_Σ} is boundedly invertible.*
4. *There exists an invertible matrix V_∞ such that*

$$D^* J_{k_+,k_-} D = V_\infty^* J_{m-k_-,k_-} V_\infty, \tag{3.20}$$

and the Riccati equation associated with the PS-Popov triple

$$\Sigma = (T(\cdot), B, \begin{bmatrix} C^*JC & C^*JD \\ D^*JC & D^*JD \end{bmatrix}),\ (J = J_{k_+,k_-}) \tag{3.21}$$

has a stabilizing solution P.

In this case, a J-spectral factor for the Popov function is given by

$$V(s) = V_\infty + L(sI - A^V)^{-1}B,$$

where $L = -JV_\infty^{-*}(D^*JC + B^*P)$, and V_∞^{-*} is the inverse of V_∞^*.

The equivalence between the first three items is a consequence of Theorem 3.9. For the equivalence between the items 1. and 4. see the proof of the Theorem 5.2 in Weiss [77]. Note that this equivalence also holds for infinite-dimensional input and output spaces.

3.3 An algorithm for computing the J-spectral factor

In this section we provide an algorithm for computing the J-spectral factor for the decomposing Banach algebra $\hat{\mathcal{W}}$.

Let us denote by P the projection of $\hat{\mathcal{W}}$ onto $\hat{\mathcal{A}}$ parallel to $\hat{\mathcal{A}}_0^\sim$ and we set $Q = I - P$. The following result gives equivalent conditions for the existence of a canonical factorization for $C \in \hat{\mathcal{W}}$.

Theorem 3.18 *Let $C \in \hat{\mathcal{W}}$. The following statements are equivalent:*

1. *The element $C = I - A$ admits a canonical factorization*

$$C = C_- C_+.$$

2. *Each of the equations*

$$\begin{aligned} X - P(AX) &= I \\ Y - Q(YA) &= I \end{aligned}$$

is solvable in $\hat{\mathcal{W}}$

3. *For any pair of elements $F, G \in \hat{\mathcal{W}}$ each of the equations*

$$\begin{aligned} X - P(AX) &= F \\ Y - Q(YA) &= G \end{aligned}$$

is uniquely solvable in $\hat{\mathcal{W}}$.

Proof: The complete proof can be found in Clancey and Gohberg [13], page 35-37. From this proof we present in detail the implication 2. $\Rightarrow$ 1. Note that by 3. we have that the equations in 2. are uniquely solvable.

Let the elements X_a and Y_a be solutions to the equations

$$X - P(AX) = I \qquad (3.22)$$

$$Y - Q(YA) = I. \qquad (3.23)$$

Defining
$$U = X_a - I \text{ and } V = Y_a - I, \tag{3.24}$$
we obtain the following relations for U and V
$$\begin{aligned} U + I - P(A(U+I)) &= I \\ V + I - Q((V+I)A) &= I. \end{aligned}$$
Thus $U \in \hat{\mathcal{A}}$ and $V \in \hat{\mathcal{A}}_0^{\sim}$. If we define U_- by $U_- = -Q(AX_a) \in \hat{\mathcal{A}}_0^{\sim}$, and V_+ by $V_+ = -P(Y_a A) \in \hat{\mathcal{A}}$, then we have
$$\begin{aligned} CX_a &= I + U_- \\ Y_a C &= I + V_+, \end{aligned}$$
since
$$\begin{aligned} CX_a - I &= (I - A)X_a - I = X_a - AX_a - I \\ &= X_a - P(AX_a) - I - Q(AX_a) \overset{(3.22)}{=} -Q(AX_a) \\ Y_a C - I &= Y_a(I - A) - I = Y_a - Y_a A - I \\ &= Y_a - Q(Y_a A) - I - P(Y_a A) \overset{(3.23)}{=} -P(Y_a A). \end{aligned}$$
Thus the following equalities hold
$$Y_a C X_a = Y_a(I + U_-) = (I + V_+)X_a.$$
So, replacing Y_a and X_a from (3.24) in the last two expressions, we get
$$Y_a C X_a = (I + V)(I + U_-) = (I + V_+)(I + U). \tag{3.25}$$
Keeping only the second equality, we see that this is equivalent to
$$V + U_- + VU_- = V_+ + U + V_+ U.$$
Since $U, V_+ \in \hat{\mathcal{A}}$ and $V, U_- \in \mathcal{A}_0^{\sim}$, this implies that $V + U_- + VU_- \in \mathcal{A}_0^{\sim}$ and $V_+ + U + V_+ U \in \hat{\mathcal{A}}$. Using the fact that $\hat{\mathcal{W}}$ is a decomposing Banach algebra we get
$$\begin{aligned} V + U_- + VU_- &= 0 \\ V_+ + U + V_+ U &= 0, \end{aligned}$$
which are the same as
$$(I + V)(I + U_-) = I \tag{3.26}$$
$$(I + V_+)(I + U) = I. \tag{3.27}$$
In particular $\det(I + V)$ and $\det(I + U)$ are nonzero, which show that $I + V$ and $I + U$ are invertible. Moreover, we have that
$$(I + V)^{-1} = I + U_- \tag{3.28}$$
$$(I + U)^{-1} = I + V_+. \tag{3.29}$$

Using (3.25) or (3.26) we obtain

$$Y_a C X_a = I.$$

Replacing Y_a and X_a, we have from (3.24) that

$$(I+V)\,C\,(I+U) = I.$$

Finally, using (3.28) and (3.29) we conclude that

$$C = (I+V)^{-1}\,(I+U)^{-1} = (I+U_-)\,(I+V_+) = Y_a^{-1}X_a^{-1}.$$

where $U, V_+ \in \hat{\mathcal{A}}$ and $V, U_- \in \hat{\mathcal{A}}_0^{\sim}$. ■

Remark 3.19 *We have the following relations between the elements defined in the proof of Theorem 3.18*

$$\begin{aligned} X_a &= I+U = (I+V_+)^{-1} \\ Y_a &= I+V = (I+U_-)^{-1} \\ C_- &= Y_a^{-1} \\ C_+ &= X_a^{-1}. \end{aligned}$$

We also have that

$$\begin{aligned} AX_a &= P(AX_a) + Q(AX_a) \\ &= U - U_- \end{aligned}$$

and

$$Y_a A = V - V_+.$$

As a corollary of the previous theorem, sufficient conditions for the existence of a canonical factorization can be given.

Corollary 3.20 *If an element $C \in G\hat{\mathcal{W}}$ satisfies*

$$\|I - C\| < \min\left\{\|P\|^{-1}, \|Q\|^{-1}\right\}, \tag{3.30}$$

where $\|P\|$, and $\|Q\|$ denote the operator norm of the projections P and Q, respectively, then C admits a canonical factorization. Moreover, whenever the inequality (3.30) holds, the factors $C_\pm$ in the canonical factorization $C = C_-C_+$ of C may be chosen as $C_+ = X_I^{-1}$, where the X_I is the solution of the equation

$$Q(X) + P(CX) = I \tag{3.31}$$

and $C_- = Y_I^{-1}$, where Y_I is the solution of

$$P(Y) + Q(YC) = I. \tag{3.32}$$

Proof: Let us consider the following operators defined on the decomposing Banach algebra $\hat{\mathcal{W}}$

$$\begin{aligned} T_C(X) &= P(CX) + Q(X), \\ R_C(Y) &= P(Y) + Q(YC). \end{aligned}$$

We see that

$$\begin{aligned} T_C(X) &= P(CX) - P(X) + X = X - P((I - C)X), \\ R_C(Y) &= Y - Q(Y) + Q(YC) = Y - Q(Y(I - C)). \end{aligned}$$

From Theorem 3.18, we have that the element C admits a canonical factorization if and only if both T_C and R_C are invertible operators on $\hat{\mathcal{W}}$. Using (3.30), we obtain that

$$\begin{aligned} \|I - T_C\| &= \|P\|\|I - C\| < 1, \\ \|I - R_C\| &= \|Q\|\|I - C\| < 1. \end{aligned}$$

Thus T_C and R_C are invertible operators on $\hat{\mathcal{W}}$ and, consequently, C admits a canonical factorization $C = C_-C_+$. From Remark 3.19 it follows that the factors $C_\pm$ may be chosen as $C_+ = X_I^{-1}$ and $C_- = Y_I^{-1}$, where X_I and Y_I are the solutions of the equations

$$\begin{aligned} X - P(CX) &= I, \\ Y - Q(YC) &= I, \end{aligned}$$

These solutions coincide with X_I and Y_I being the solutions of the equations (3.31) and (3.32), respectively. ■

Remark 3.21 *If we look more closely at the relations (3.31) and (3.32), then one can observe that*

1. *The equation (3.31) is nothing other that $Q(X) = I - P(CX)$. This is equivalent to the equations*

$$\begin{cases} Q(X) &= 0 \\ P(CX) &= I. \end{cases}$$

   The first equation means that $X \in \hat{\mathcal{A}}$, and the second equation that $I - CX \in \hat{\mathcal{A}}^\sim$.

2. *The equation (3.32) is nothing other that $Q(YC) = I - P(Y)$. This is equivalent to the equations*

$$\begin{cases} Q(YC) &= 0 \\ P(Y) &= I. \end{cases}$$

   The first equation means that $YC \in \hat{\mathcal{A}}$, and the second equation that $Y - I \in \hat{\mathcal{A}}_0^\sim$.

Before presenting an algorithm for constructing the J-spectral factor we need to state the following lemma.

Lemma 3.22 *Let $C_+ \in G\hat{\mathcal{A}}$ and R a rational matrix-valued function that is invertible over RL_∞. There exist matrix-valued functions R_1 that is invertible over RL_∞ and $B_+ \in G\hat{\mathcal{A}}$ such that*

$$C_+R = R_1B_+. \tag{3.33}$$

Proof: The proof is a constructive one. Let q be a polynomial which has as zeroes the roots of all the denominators of R with positive real part and the corresponding multiplicities, and let p be a polynomial of the same degree as q only having zeroes with the negative real part. Consequently, we can write

$$R = \frac{p}{q}R_+,$$

where $\frac{p}{q}$ is a scalar and $R_+ \in \hat{\mathcal{A}}$. We have that $C_+R_+ \in \hat{\mathcal{A}}$ and it has a nonzero determinant over the imaginary axis including infinity. Since $C_+ \in G\hat{\mathcal{A}}$, there are only a finite number s_1, s_2, ..., s_n of zeros of $\det(C_+(s)R_+(s))$ in the right half-plane, counted with respect to multiplicity. In fact they are the unstable zeroes of $\det(R_+(s))$. Let

$$C_+R_+ = \begin{bmatrix} l_1 \\ l_2 \\ . \\ . \\ l_n \end{bmatrix}$$

be a representation of C_+R_+ in terms of row vectors. The vectors $l_1(s_1)$, $l_2(s_1)$, ..., $l_n(s_1)$ are linearly dependent. Thus there exist some α_1, α_2, ... α_n not all zero (let assume for example that $\alpha_p \neq 0$) such that

$$\alpha_1 l_1(s_1) + \alpha_2 l_2(s_1) + ... + \alpha_n l_n(s_1) = 0.$$

We make the following notation

$$l_p'(s) = \frac{s+s_1}{s-s_1}\left[\alpha_1 l_1(s) + ... + \alpha_n l_n(s)\right].$$

We have that

$$C_+R_+ = E_1 \begin{bmatrix} l_1 \\ . \\ . \\ l_p' \\ . \\ . \\ l_n \end{bmatrix},$$

where $E_1 \in RL_\infty$ is the following one

$$\begin{bmatrix} 1 & 0 & . & . & 0 & . & . & 0 \\ 0 & 1 & & & 0 & . & . & 0 \\ . & 0 & . & & . & & & . \\ . & & & . & & & & \\ -\frac{\alpha_1}{\alpha_p} & . & . & . & \frac{s-s_1}{s+s_1} & . & . & -\frac{\alpha_n}{\alpha_p} \\ . & & & & . & & & . \\ . & & & & . & & & . \\ 0 & . & & & 0 & & . & 1 \end{bmatrix}.$$

We see that E_1 is invertible over RL_∞. The matrix

$$\begin{bmatrix} l_1 \\ . \\ . \\ l'_p \\ . \\ . \\ l_n \end{bmatrix}$$

belongs to $\hat{\mathcal{A}}$ and its determinant has only the zeros s_2, ...,s_n in the open right half-plane. Continuing in this manner we obtain that

$$C_+R_+ = E_1E_2...E_nB_+.$$

We note that $B_+ \in \hat{\mathcal{A}}$. By construction, the right half-plane zeros of the functions $\det(C_+(s)R_+(s))$ and $\det(E_1(s)E_2(s)...E_n(s))$ are the same. From the last equality we conclude that $\det B_+(s)$ has no zeros in the right half-plane. Moreover, $B_+(s)$ is given by

$$B_+ = (E_1E_2...E_n)^{-1}C_+R_+,$$

so, from Proposition 2.20 we can conclude that $B_+ \in G\hat{\mathcal{A}}$. Finally, we have that

$$C_+R = R_1B_+,$$

where

$$R_1 = \frac{p}{q}E_1E_2...E_n. \tag{3.34}$$

■

In the following we will present an algorithm for computing the J-spectral factor for a matrix-valued function $Z \in \hat{\mathcal{W}}$, provided it exists.

Let $Z = Z^\sim \in \hat{\mathcal{W}}$ be invertible over $\hat{\mathcal{W}}$.

Step 1. We can write

$$Z = CR \tag{3.35}$$

for any nonsingular $R \in RL_\infty$, where

$$C = I - (R - Z)R^{-1}. \tag{3.36}$$

Due to the fact that $\hat{\mathcal{W}}$ is an R-algebra, there exists a rational matrix-valued function R invertible over RL_∞ such that

$$\begin{aligned} \|R - Z\| &< \epsilon_1, \\ \|R^{-1}\| &< \|Z^{-1}\| + \epsilon_2. \end{aligned}$$

Thus we can choose R such that

$$\|(R - Z)R^{-1}\|_\infty < \min\{\|P\|^{-1}, \|Q\|^{-1}\}.$$

Step 2. Using Corollary 3.20, there exists the canonical factorization

$$C = I - (R - Z)R^{-1} = C_- C_+. \tag{3.38}$$

This factorization can be obtained solving the following two operator equations for X and Y

$$Q(X) + P(CX) = I \tag{3.39}$$

$$P(Y) + Q(YC) = I. \tag{3.40}$$

If we name the solutions X_I and Y_I, respectively, we have that

$$\begin{aligned} C_- &= Y_I^{-1}, \\ C_+ &= X_I^{-1}. \end{aligned} \tag{3.41}$$

Step 3. As in Lemma 3.22 we can write

$$C_- C_+ R = C_- R_1 B_+,$$

where $R_1 \in RL_\infty$ and $B_+ \in G\hat{\mathcal{A}}$. Note that R_1 and B_+ can be obtained in a constructive manner.

Step 4. Since $\det R_1(s) \neq 0$ on the imaginary axis (see (3.34) for the definition of R_1), from Theorem 3.2 we can conclude that there exists a standard factorization

$$R_1 = R_- D R_+,$$

where $R_- \in \hat{\mathcal{A}}^\sim$ and $R_+ \in \hat{\mathcal{A}}$. We have now that

$$Z = C_- R_- D R_+ B_+.$$

Step 5. Since we know that there exist a J-spectral factorization of the matrix-valued function $Z \in \hat{\mathcal{W}}$ it follows that Z has no equalizing vectors, so D is the identity (see Theorem 3.9 and its proof).

$$Z = Z_- Z_+,$$

where

$$Z_- = C_- R_- \tag{3.42}$$
$$Z_+ = R_+ B_+ . \tag{3.43}$$

Step 6. We write

$$Z_-^\sim Z_+^{-1} = U^\sim J U.$$

This is possible because we have

$$Z_- Z_+ = Z = Z^\sim = Z_+^\sim Z_-^\sim,$$

which can be rewritten as

$$(Z_+^\sim)^{-1} Z_- = Z_-^\sim Z_+^{-1}. \tag{3.44}$$

The left-hand side is an element of $\hat{\mathcal{A}}^\sim$ and the right-hand side is an element of $\hat{\mathcal{A}}$. Thus $Z_-^\sim Z_+^{-1}$ is a constant matrix, which we denote by M. The equation (3.44) shows $M = M^*$. Since $Z_-^\sim$ and Z_+^{-1} are invertible, $\det(M) \neq 0$. Thus there exists an unitary matrix U such that

$$M = U^\sim J U. \tag{3.45}$$

Step 7. We obtained the J-spectral factorization

$$Z = V^\sim J V, \tag{3.46}$$

where

$$V = U Z_+.$$

This can be shown as follows. Using once again (3.44) we get that $Z_- = Z_+^\sim U^\sim J U$, and thus $Z = Z_+^\sim U^\sim J U Z_+$. The choice $V = U Z_+$ is the right one for the J-spectral factor of the matrix-valued function Z.

We summarize now the steps described before in the following algorithm.

Algorithm 3.23 *The following seven steps describe a procedure how to obtain the J-spectral factor for a matrix-valued function $Z \in G\hat{\mathcal{W}}$ provided such a factor exists.*

Step 1. *Write $Z = CR$ with R an matrix-valued function invertible over RL_∞, such that*

$$\|(R - Z)R^{-1}\|_\infty < \min\{\|P\|^{-1}, \|Q\|^{-1}\}.$$

Step 2. *Solve the equation (3.39) and (3.40), and write*

$$Z = C_- C_+ R$$

where C_- and C_+ are given by (3.41).

Step 3. *$Z = C_- R_1 B_+$ (see Lemma 3.22).*

Step 4. *$Z = C_- R_- D R_+ B_+$.*

Step 5. *If that a Z has no equalizing vectors then admits a J-spectral factorization. We write*

$$Z = Z_- Z_+,$$

where Z_- and Z_+ are given by (3.42) and (3.43) respectively.

Step 6. *Factorize the constant matrix $Z_-^{\sim} Z_+^{-1}$:*

$$Z_-^{\sim} Z_+^{-1} = U^{\sim} J U.$$

Step 7. *Write the J-spectral factorization*

$$Z = V^{\sim} J V,$$

where

$$V = U Z_+ = U R_+ B_+.$$

From Theorem 3.4 we have the form of all J-spectral factors, provided that a J-spectral factorization exists.

Chapter 4

Nehari problems and equalizing vectors

Introduction

The Nehari problem is naturally formulated in the frequency-domain: given a matrix-valued function $G \in L_\infty$, find the distance from G to the stable matrix-valued functions. The problem of finding K that achieves the minimum distance is called the Nehari extension problem. In this chapter we consider the Nehari extension problem together with a special version of this problem, known as the sub-optimal Nehari extension problem. This is: given a matrix-valued function $G \in L_\infty$ and a $\sigma > 0$, find (if it exists) a stable K such that

$$\|G + K\|_\infty = \operatorname*{ess\,sup}_{s \in \mathbb{C}_0} \|G(s) + K(s)\| < \sigma.$$

For a class of infinite-dimensional systems we obtain an elementary frequency-domain solution for the sub-optimal Nehari extension problem. The approach is via J-spectral factorization, and it uses the concept of an equalizing vector introduced in the previous chapter. The connection between the equalizing vectors and the Nehari extension problem is given.

These problems have received wide attention in the mathematical systems and control literature (see Adamjan et al. [2], Ball and Halton [6], Curtain and Ran [19], Curtain and Zwart [22], Foias and Tannenbaum [26], Green et al. [33], Ran [59], Sasane and Curtain [65], [67]). Several control problems can be reduced to a Nehari problem (see e.g. Curtain and Zwart [23], chapter 9). In Curtain and Green [15], the sub-optimal Nehari extension problem is used, in an essential way, for solving the standard H_∞ sub-optimal control problem for a class of infinite-dimensional systems. For the solution of the Nehari extension problem, the authors of [15] refer to the abstract results in Ball and Helton [5], [6]. If one looks in the papers [5], [6], the result stated in [15] is not an obvious corollary of the abstract results presented by Ball and Helton. This motivated our investigation.

For the Wiener class of infinite-dimensional systems we give a direct frequency-domain solution for the sub-optimal Nehari extension problem. The approach is via J-spectral factorization, and uses results presented in Chapter 2. Via a simple proof, we show that the sub-optimal Nehari extension problem is solvable if and only if a certain J-spectral factorization exists. The simple proof is based on the concept of equalizing vectors, as defined in Chapter 3. This concept was introduced, for finite-dimensional systems, by Meinsma [50].

In the first section of this chapter some preliminary results are presented. The sub-optimal Nehari extension problem is solved in the sesond section. The connection between the equalizing vectors and the Nehari extension problem is provided in the third section of this chapter. An expression in terms of a state space representation for Pritchard-Salamon systems is given in the last section.

4.1 Preliminaries

In this section we prove some useful lemmas.
Using Corollary 2.21, a similar result as in Lemma 2.10 can be stated for the Wiener algebra.

Lemma 4.1 *Let P_{21}, P_{22}, Q_1 and Q_2 be stable ($\in \hat{\mathcal{A}}$) matrix-valued functions of appropriate dimensions. Suppose that P_{22} and Q_2 are invertible in L_∞ and that the inequalities*

$$\begin{aligned} \|P_{22}^{-1}P_{21}\|_{L_\infty} &\leq 1, \\ \|Q_1 Q_2^{-1}\|_{L_\infty} &< 1, \end{aligned}$$

are satisfied. Then the following two statements are equivalent

1. *P_{22} and Q_2 are bistable;*
2. *$P_{21}Q_1 + P_{22}Q_2$ is bistable.*

It is important to note that in the following lemma J_{γ,n_w,n_z} and J_{n_y,n_z} have the same number (n_z) of negative eigenvalues.

Lemma 4.2 *Let $P \in L_\infty^{(n_w+n_z)\times(n_y+n_z)}$, and suppose that*

$$P^{\sim}(s)J_{\gamma,n_w,n_z}P(s) = J_{n_y,n_z} \tag{4.1}$$

almost everywhere on the imaginary axis. Consider the equality

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{4.2}$$

with $X_2 \in H_\infty^{n_z\times n_z}$, $Q_1 \in H_\infty^{n_y\times n_z}$, $Q_2 \in H_\infty^{n_z\times n_z}$, $P_{21} \in H_\infty^{n_z\times n_y}$, $P_{22} \in H_\infty^{n_z\times n_z}$. Then the following two conditions are equivalent

1. *X_2 is bistable and* $\left\|X_1X_2^{-1}\right\|_{L_\infty} < \gamma$,
2. *P_{22} and Q_2 are bistable and* $\|Q_1Q_2^{-1}\|_{H_\infty} < 1$.

Proof: We will prove this lemma in three steps. We make use of Lemma 2.5, which states that a matrix-valued function $F \in L_\infty$ is invertible in L_∞ if and only if $F^\sim F > \varepsilon I$ almost everywhere on the imaginary axis for some strictly positive ε.

Step 1. First we prove that the following two statements are equivalent

a. X_2 is bistable and $\|X_1X_2^{-1}\|_{L_\infty} < \gamma$

b. $P_{21}Q_1 + P_{22}Q_2$ is bistable, Q_2 is invertible in L_∞ and $\|Q_1Q_2^{-1}\|_{L_\infty} < 1$

From the equality (4.2) we see that

$$X_2 = P_{21}Q_1 + P_{22}Q_2,$$

which means that bistability of X_2 is the same as the bistability of $P_{21}Q_1 + P_{22}Q_2$. Moreover, we have the following sequence of equivalences

$$\begin{aligned}
\|X_1X_2^{-1}\|_{L_\infty} < \gamma \quad & \Leftrightarrow \quad (X_1X_2^{-1})^\sim X_1X_2^{-1} - \gamma^2 I < -\varepsilon I < 0 \\
& \qquad \text{almost everywhere on the imaginary axis,} \\
& \qquad \text{for some } \varepsilon > 0 \\
& \Leftrightarrow \quad X_1^\sim X_1 - \gamma^2 X_2^\sim X_2 < -\delta I < 0 \\
& \qquad \text{almost everywhere on the imaginary axis,} \\
& \qquad \text{for some } \delta > 0 \\
& \Leftrightarrow \quad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^\sim J_\gamma \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} < -\delta I < 0 \\
& \qquad \text{almost everywhere on the imaginary axis,} \\
& \qquad \text{for some } \delta > 0 \\
& \overset{(4.2),(4.1)}{\Leftrightarrow} \quad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^\sim J \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} < -\delta I < 0 \\
& \qquad \text{almost everywhere on the imaginary axis,} \\
& \qquad \text{for some } \delta > 0 \\
& \Leftrightarrow \quad Q_1^\sim Q_1 - Q_2^\sim Q_2 < -\delta I < 0 \\
& \qquad \text{almost everywhere on the imaginary axis,} \\
& \qquad \text{for some } \delta > 0 \\
& \Leftrightarrow \quad Q_2 \text{ is invertible over } L_\infty \text{ and} \\
& \qquad \|Q_1Q_2^{-1}\|_{L_\infty} < 1.
\end{aligned}$$

Step 2. We show that

a. P_{22} is invertible over L_∞ and

b. $\|P_{22}^{-1}P_{21}\|_{L_\infty} \leq 1$.

From the relation (4.1) we have that almost everywhere on the imaginary axis the following equality holds

$$P_{12}^{\sim}P_{21} - \gamma^2 P_{22}^{\sim}P_{22} = -I,$$

which implies that

$$P_{22}^{\sim}P_{22} > \alpha I$$

almost everywhere on the imaginary axis, for some $\alpha < \frac{1}{\gamma^2}$. Applying now Lemma 2.5, we conclude that P_{22} is invertible over L_∞. So, the first statement is established.

In order to prove the second statement, we consider the input-output equality determined by the plant P

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

with $y_1, y_2, u_1, u_2 \in L_2(\mathbb{C}_0)$. Using relation (4.1), we have

$$\begin{aligned}
\|y_1\|_2^2 - \gamma^2\|y_2\|_2^2 &= \int_{-\infty}^{\infty} (\|y_1(j\omega)\|^2 - \gamma^2\|y_2(j\omega)\|^2)d\omega \\
&= \int_{-\infty}^{\infty} (y_1(j\omega)^{\sim}y_1(j\omega) - \gamma^2 y_2(j\omega)^{\sim}y_2(j\omega))d\omega \\
&= \int_{-\infty}^{\infty} (\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^{\sim} J_\gamma \begin{bmatrix} y_1 \\ y_2 \end{bmatrix})d\omega \\
&= \int_{-\infty}^{\infty} (\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{\sim} P^{\sim}J_\gamma P \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})d\omega \\
&= \int_{-\infty}^{\infty} (\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{\sim} J \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})d\omega \\
&= \int_{-\infty}^{\infty} (\|u_1(j\omega)\|^2 - \|u_2(j\omega)\|^2)d\omega \\
&= \|u_1\|_2^2 - \|u_2\|_2^2.
\end{aligned}$$

For any $u_1 \in L_2(\mathbb{C}_0)$, we define

$$u_2 = -P_{22}^{-1}P_{21}u_1.$$

Note that since P_{22} is invertible over L_∞, we have that $u_2 \in L_2(\mathbb{C}_0)$. For this choice of u_2 the output y_2 is

$$y_2 = P_{21}u_1 + P_{22}u_2 = P_{21}u_1 - P_{22}P_{22}^{-1}P_{21}u_1 = 0,$$

and

$$\|y_1\|_2^2 = \|y_1\|_2^2 - \gamma^2\|y_2\|_2^2 = \|u_1\|_2^2 - \|P_{22}^{-1}P_{21}u_1\|_2^2.$$

This implies that

$$\frac{\|P_{22}^{-1}P_{21}u_1\|_2^2}{\|u_1\|_2^2} \leq 1.$$

Since this holds for all $u_1 \in L_2(\mathbb{C}_0)$, we conclude that the second assertion holds.

Step 3. Now, we prove the equivalence between

1. X_2 is bistable and $\left\|X_1X_2^{-1}\right\|_{L_\infty} < \gamma$
2. P_{22} and Q_2 are bistable and $\|Q_1Q_2^{-1}\|_{H_\infty} < 1$.

Since the results in Step 2 hold, we can use Lemma 4.1 to see that the following two conditions are equivalent

a. $P_{21}Q_1 + P_{22}Q_2$ bistable, Q_2 is invertible in L_∞ and $\|Q_1Q_2^{-1}\|_{L_\infty} < 1$,

b. P_{22} and Q_2 are bistable and $\|Q_1Q_2^{-1}\|_{H_\infty} < 1$.

Combining this equivalence with the equivalence proved in Step 1, we have completed the proof. ∎

Remark 4.3 *The rational case for Lemma 4.1 and Lemma 4.2 were proved in Vidyasagar [74] (pp. 274-275) and Green et al. [33], respectively. The proof presented here follows the lines of a similar result obtained for $P \in H_\infty$ by Meinsma and Zwart [51] (see Lemma 6.1 and Theorem 6.2).*

Using Lemma 2.23 and Lemma 4.1 we can state a similar result for the decomposing Banach algebra $\hat{\mathcal{W}}$.

Lemma 4.4 *Let $P \in \hat{\mathcal{W}}^{(n_w+n_z)\times(n_y+n_z)}$, and suppose that*

$$P^{\sim}(s)\, J_{\gamma,n_w,n_z} P(s) = J_{n_y,n_z}, \text{ for all } s \in \mathbb{C}_0. \tag{4.3}$$

Consider the equality

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{4.4}$$

with $X_2 \in \hat{\mathcal{A}}^{n_z\times n_z}$, $Q_1 \in \hat{\mathcal{A}}^{n_y\times n_z}$, $Q_2 \in \hat{\mathcal{A}}^{n_z\times n_z}$, $P_{21} \in \hat{\mathcal{A}}^{n_z\times n_y}$, $P_{22} \in \hat{\mathcal{A}}^{n_z\times n_z}$. Then the following two conditions are equivalent

1. *X_2 is bistable and $\left\|X_1X_2^{-1}\right\|_{L_\infty} < \gamma$*
2. *P_{22} and Q_2 are bistable and $\left\|Q_1Q_2^{-1}\right\|_{H_\infty} < 1$*

A transposed version of Lemma 4.2 can be easily stated.

Lemma 4.5 *Let $M \in L_\infty^{(n_y+n_z)\times(n_w+n_z)}$ with $n_y = n_w$, and suppose that*

$$M(s)J_{\gamma,n_y,n_z}M^{\sim}(s) = J_{n_w,n_z},\ \textit{for almost all } s \in \mathbb{C}_0. \tag{4.5}$$

Consider the equality

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{4.6}$$

with $H_1 \in H_\infty^{n_z\times n_w}$, $H_2 \in H_\infty^{n_z\times n_z}$, $U_1 \in H_\infty^{n_z\times n_y}$, $U_2 \in H_\infty^{n_z\times n_z}$, $M_{12} \in H_\infty^{n_y\times n_z}$, and $M_{22} \in H_\infty^{n_z\times n_z}$. Then the following two conditions are equivalent

1. *H_2 is bistable and $\left\|H_2^{-1}H_1\right\|_{H_\infty} < \gamma$.*

2. *M_{22} and U_2 are bistable and $\left\|U_2^{-1}U_1\right\|_{H_\infty} < 1$.*

Proof: Taking the transpose in the relations (4.5) and (4.6), we obtain (4.1) and (4.2) where $P = M^T$, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \end{bmatrix}^T$, $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$. Using Lemma 2.9, we have the following equivalent statements

$$\begin{aligned} &H_2 \text{ is bistable and } \left\|H_2^{-1}H_1\right\|_{H_\infty} < \gamma \Leftrightarrow \\ &H_2^T \text{ is bistable and} \left\|\left(H_2^{-1}H_1\right)^T\right\|_{H_\infty} < \gamma \Leftrightarrow \\ &H_2^T \text{ is bistable and} \left\|H_1^T\left(H_2^T\right)^{-1}\right\|_{H_\infty} < \gamma \ \Leftrightarrow \\ &X_2 \text{ is bistable and } \left\|X_1X_2^{-1}\right\|_{H_\infty} < \gamma. \end{aligned} \tag{4.7}$$

Similarly, it follows that

$$\begin{aligned} &M_{22} \text{ and } U_2 \text{ are bistable and } \left\|U_2^{-1}U_1\right\|_{H_\infty} < 1 \Leftrightarrow \\ &\Leftrightarrow P_{22} \text{ and } Q_2 \text{ are bistable and } \left\|Q_1Q_2^{-1}\right\|_{H_\infty} < 1. \end{aligned} \tag{4.8}$$

Now using Lemma 4.4 toghether with (4.7) and (4.8) we have proved the equivalence between 1. and 2. ■

A similar result holds also for $M \in \hat{\mathcal{W}}$.

The Nehari problem admits an elegant solution in terms of the Hankel operator of a matrix-valued function G. It is well known that L_2^n is the direct sum of H_2^n and $H_2^{n,\perp}$ with respect to the usual inner product. We introduce the frequency-domain *Hankel operator* with symbol $G \in L_\infty^{k\times m}$, defined by

$$H_G : H_2 \to H_2^\perp\ ,\ H_G u = \Pi_- G u \text{ for } u \in H_2. \tag{4.9}$$

Its adjoint is

$$H_G^* : H_2^\perp \to H_2\ ,\ H_G^* v = \Pi_+ G^{\sim} v \text{ for } v \in H_2^\perp. \tag{4.10}$$

Here Π_+ and Π_- are the orthogonal projection from L_2 to H_2 and $H_2^\perp$, respectively (see [27]).

As in Curtain and Zwart [23] (see Lemma 8.1.7, page 390), the following lemma gives sufficient conditions for the Hankel operator to be compact.

Lemma 4.6 *If $G = G_1 + G_2$, where $G_1 \in H_\infty$ and $G_2 \in L_\infty$ is continuous on the imaginary axis with zero limit at infinity, then the Hankel operator with symbol G is compact.*

If the Hankel operator with symbol $G \in L_\infty^{k\times m}$ is compact, then we denote the singular values of H_G (that is, the nonnegative square roots of the eigenvalues of $H_G^* H_G$), by $\sigma_1 \geq \sigma_2 \geq \ldots (\geq 0)$. The σ_k's are than referred to as the *Hankel singular values of* G. We remark that the Hankel operator with symbol $G \in \hat{\mathcal{W}}^{k\times m}$ is compact (see the above lemma or Lemma 8.2.4. and Lemma 8.2.6. in Curtain and Zwart [23]).

4.2 The sub-optimal Nehari extension problem

Using the fact that the sub-optimal Nehari extension problem is trivial for stable matrix-valued functions, we can restrict this problem, without loss of generality, to antistable matrix-valued functions. The following theorem is our main result. The finite dimensional version of this theorem is equivalent to a result of Kimura [44], Theorem 7.4, published earlier by Green et al. in [33].

Theorem 4.7 *Let G be a matrix-valued function such that $G^{\sim} \in \hat{\mathcal{A}}^{k\times m}$, and σ a positive real number. The following statements are equivalent:*

1. $\|H_G\| < \sigma$.
2. *There exists $K(s) \in \hat{\mathcal{A}}^{k\times m}$ such that*
$$\|G + K\|_\infty < \sigma. \tag{4.11}$$
3. *There exists a J-spectral factor $\Lambda(s) \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ for*
$$W(s) = \begin{bmatrix} I_k & 0 \\ G^{\sim}(s) & I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix} \tag{4.12}$$
with $\Lambda_{11}^{-1}(s) \in \hat{\mathcal{A}}^{k\times k}$

Furthermore, all solutions for the sub-optimal Nehari extension problem are parametrized by
$$K(s) = X_1(s) X_2(s)^{-1},$$
where
$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix}, \tag{4.13}$$
with $Q(s) \in \hat{\mathcal{A}}^{k\times m}$, $\|Q\|_\infty < 1$.

Remark 4.8 *The equivalence between the first two items is well-known (see for example Theorem 8.3.9 in Curtain and Zwart [23]). We will present only the proof of the equivalence between the items 2. and 3.*

In third statement of Theorem 4.7 we get an extra condition for the J-spectral factor, namely Λ_{11}^{-1} is a stable matrix-valued function. The following lemma proves that this property is invariant with respect to the J-spectral factors.

Lemma 4.9 *Let W be given as in (4.12), and suppose that X is a J-spectral factor for W such that X_{11}^{-1} is a stable matrix-valued function. If Y is another J-spectral factor for Z, then we have that Y_{11}^{-1} is also a stable matrix-valued function.*

Proof: Since X is a J-spectral factor for W given in (4.12), we have that

$$X^{\sim}(s)JX(s) = W(s)$$

on the imaginary axis. This means that also the equality

$$X_{11}^{\sim}(s)X_{11}(s) - X_{21}^{\sim}(s)X_{21}(s) = I$$

holds on the imaginary axis. Together with the fact that X and X_{11}^{-1} are stable, this implies that

$$\|X_{21}X_{11}^{-1}\|_{H_\infty} \leq 1.$$

Let $Q = Y(s)X(s)^{-1}$. From Theorem 3.4, we see that Q is a constant matrix satisfying

$$Q^*JQ = J$$

and so

$$Q^{-1} = J^{-1}Q^*J \text{ and } QJ^{-1}Q^* = J^{-1}.$$

The $(1,1)$ block of the last equality yields

$$Q_{11}Q_{11}^* - Q_{12}Q_{12}^* = I,$$

or equivalently

$$\|Q_{11}^*x\|^2 - \|Q_{12}^*x\|^2 = \|x\|^2$$

for $x \in \mathbb{C}^k$. So $\det Q_{11}^* \neq 0$ and

$$\|y\|^2 - \|(Q_{11}^*)^{-1}y\|^2 = \|Q_{12}^*(Q_{11}^*)^{-1}y\|^2$$

for $y \in \mathbb{C}^k$. Thus

$$\|Q_{12}^*(Q_{11}^*)^{-1}\| < 1.$$

Since $\|R\| = \|R^*\|$, we have that

$$\|Q_{11}^{-1}Q_{12}\| < 1.$$

From the last inequality and the fact that $\|X_{21}X_{11}^{-1}\|_{H_\infty} \leq 1$, we obtain that

$$\|Q_{11}^{-1}Q_{12}X_{21}X_{11}^{-1}\|_{H_\infty} < 1.$$

Since $Q_{11}^{-1}Q_{12}X_{21}X_{11}^{-1}$ is stable, from Lemma 2.8 we have that $I+Q_{11}^{-1}Q_{12}X_{21}X_{11}^{-1}$ is bistable, and so

$$\begin{aligned} Y_{11} &= Q_{11}X_{11}+Q_{12}X_{21} \\ &= Q_{11}(I+Q_{11}^{-1}Q_{12}X_{21}X_{11}^{-1})X_{11} \end{aligned}$$

is bistable. ■

Before we give the proof of the main theorem we prove a result which provides the existence of a J-spectral factorization.

Theorem 4.10 *Let $G \in \hat{\mathcal{W}}^{k\times m}$ be a matrix-valued function of a complex variable such that $G^\sim \in \hat{\mathcal{A}}^{m\times k}$ and σ a positive real number which satisfies $\sigma_{l+1} < \sigma < \sigma_l$ or $\sigma_1 < \sigma$. Then there exists a $(k+m)\times(k+m)$-matrix-valued function $\Lambda \in \hat{\mathcal{A}}$ such that $W(s)$, defined by*

$$W(s) = \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}^\sim J_{\sigma,k,m} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}, \tag{4.14}$$

has the $J_{k,m}$-spectral factorization

$$W(s) = \Lambda(s)^\sim J_{k,m}\Lambda(s). \tag{4.15}$$

Moreover, if G is strictly proper, then Λ can be chosen such that

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} \Lambda(s) = \begin{bmatrix} I_k & 0 \\ 0 & \sigma I_m \end{bmatrix}. \tag{4.16}$$

Proof: It is easy to see that $W(s) = W^\sim(s)$, and $\det W(s) \neq 0$ for all $s \in \mathbb{C}_0 \cup \{\infty\}$. In order to prove that the matrix-valued function $W(s)$ has a J-spectral factorization, it is enough to show that $W(s)$ has no equalizing vectors, see Theorem 3.9.

Let u be an equalizing vector for the matrix-valued function $W(s)$. It means that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2,\ u \neq 0,\ Wu = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in H_2^\perp. \tag{4.17}$$

So, we have that

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= Wu = \begin{bmatrix} I_k & 0 \\ G^\sim & I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} I_k & G \\ G^\sim & G^\sim G - \sigma^2 I_m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \end{aligned}$$

which is equivalent to

$$\left\{ \begin{array}{c} u_1 + Gu_2 = v_1, \\ G^\sim u_1 + G^\sim G u_2 - \sigma^2 u_2 = v_2. \end{array} \right.$$

In the first equality we split Gu_2 using the projections Π_- and Π_+. We obtain that

$$\begin{cases} u_1 + \Pi_+ G u_2 = v_1 - \Pi_- G u_2, \\ G^\sim(u_1 + G u_2) - \sigma^2 u_2 = v_2. \end{cases} \tag{4.18}$$

From (4.17) and the definition of the projection operators we have that the left-hand side of the first equality lies in H_2, whereas the right-hand side lies in $H_2^\perp$. This implies that

$$u_1 + \Pi_+ G u_2 = 0 \text{ and } v_1 - \Pi_- G u_2 = 0. \tag{4.19}$$

Using the first equality in (4.19) we replace u_1 in the second equality of (4.18) and split the term $G^\sim \Pi_- G u_2$ according to the projections. We have that

$$\begin{aligned} G^\sim \Pi_- G u_2 - \sigma^2 u_2 = v_2 \quad &\Leftrightarrow \quad \Pi_- G^\sim \Pi_- G u_2 + \Pi_+ G^\sim \Pi_- G u_2 - \sigma^2 u_2 = v_2 \\ &\Leftrightarrow \quad \Pi_+ G^\sim \Pi_- G u_2 - \sigma^2 u_2 = v_2 - \Pi_- G^\sim \Pi_- G u_2. \end{aligned}$$

Using similar arguments as before we have that

$$\Pi_+ G^\sim \Pi_- G u_2 - \sigma^2 u_2 = 0, \tag{4.20}$$

which is equivalent to $(H_G^* H_G - \sigma^2 I_m)u_2 = 0$. Since σ is not a singular value of the Hankel operator, we obtain that u_2 must be zero. From (4.19) we see that also u_1 must be zero, so $u = 0$. We conclude that the matrix-valued function W has no equalizing vectors, which by Theorem 3.9 implies that W has a J-spectral factorization (4.15).

If G is a strictly proper matrix-valued function we see that the limit of W at $\pm j\infty$ is the matrix $J_{\sigma,k,m}$. Consequently, it is easy to check that if there exists a J-spectral factor Λ_0 with the limit Λ_∞ at $\pm j\infty$, then Λ defined by

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} \Lambda_\infty^{-1} \Lambda_0(s)$$

is clearly a J-spectral factor with the limit $\begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}$ at $\pm j\infty$. From the properties of the Wiener algebras we see that (4.16) holds. ■

Remark 4.11 *The previous result can be reformulated as follows: If $\sigma > 0$ belongs to the resolvent set of the Hankel operator with symbol G, then the matrix-valued function W has a J-spectral factorization.*

We the above results we can now prove Theorem 4.7.

Proof: <u>2. ⇒ 3.</u> : From Theorem 4.10 we have that the matrix-valued function W has a J-spectral factorization (4.15). Let Λ be a J-spectral factor. We prove that $\Lambda_{11}(s)^{-1}$ is a stable matrix-valued function. The following equality holds

$$\begin{aligned} \begin{bmatrix} G+K \\ I \end{bmatrix} &= \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \Lambda^{-1} \Lambda \begin{bmatrix} K \\ I \end{bmatrix} \\ &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} &:= \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \Lambda^{-1} \text{ and} \\ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &:= \Lambda \begin{bmatrix} K \\ I \end{bmatrix}\end{aligned} \tag{4.21}$$

with P_{21}, P_{22}, Q_1 and Q_2 stable matrix-valued functions. Now, by the definition of P, the following equality holds on the imaginary axis

$$P^{\sim} J_{\sigma,k,m} P = (\Lambda^{-1})^{\sim} \begin{bmatrix} I & G \\ 0 & I \end{bmatrix}^{\sim} J_{\sigma,k,m} \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \Lambda^{-1} = J. \tag{4.22}$$

Combining this with (4.11), we conclude from Lemma 4.4 with $X_1 = G + K$, and $X_2 = I$ that P_{22} is bistable. Using matrix block manipulation, it can be checked that $V_{22} = P_{22}$ and $\Lambda_{11}^{-1} = V_{11} - V_{12} P_{22}^{-1} V_{21}$, where $V = \Lambda^{-1}$. Since all the elements expressing Λ_{11}^{-1} are stable, we have that Λ_{11}^{-1} is also stable.

Applying Lemma 4.4 once more, we obtain that Q_2 is a bistable matrix-valued function and $\|Q_1 Q_2^{-1}\|_\infty < 1$. Multiplying the relation (4.21) to the left with Λ^{-1} and to the right with Q_2^{-1} we have that

$$\begin{bmatrix} K Q_2^{-1} \\ Q_2^{-1} \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} Q_1 Q_2^{-1} \\ I \end{bmatrix}. \tag{4.23}$$

Denoting $X_1 = K Q_2^{-1}$ and $X_2 = Q_2^{-1}$, gives

$$X_1 X_2^{-1} = K Q_2^{-1} Q_2 = K$$

and, using (4.23), X_1 and X_2 satisfy (4.13), with $Q = Q_1 Q_2^{-1}$, which is stable.

<u>3. ⇒ 2.</u> : Suppose that there exists a J-spectral factor Λ for the matrix-valued function W such that Λ_{11} is bistable. Let V denote Λ^{-1}. Using matrix block manipulation (see also Kailath [43]), it can be proved that

$$V_{22}(s)^{-1} = \Lambda_{22}(s) - \Lambda_{21}(s) \Lambda_{11}(s)^{-1} \Lambda_{12}(s).$$

Since $\Lambda_{22}(s)$, $\Lambda_{21}(s)$, $\Lambda_{11}(s)^{-1}$ and $\Lambda_{12}(s)$ are stable, also $V_{22}(s)^{-1}$ is stable. So, we conclude that V_{22} is a bistable matrix-valued function. If we define $K_0 = V_{12} V_{22}^{-1}$, then K_0 is stable. Furthermore, we have the following equality

$$\begin{bmatrix} G + K_0 \\ I_m \end{bmatrix} = \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix} := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}.$$

Since Λ is a J-spectral factor for W, $P_{22} = V_{22}$, $Q_2 = V_{22}^{-1}$ and $\|Q_1 Q_2^{-1}\|_\infty = 0 < 1$, we can apply Lemma 4.4, and we see that K_0 is a solution for the sub-optimal Nehari extension problem.

Consider $K = X_1 X_2^{-1}$, where X_1 and X_2 are given by (4.13). Using

$$\begin{bmatrix} G + K \\ I_m \end{bmatrix} = \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} X_2^{-1}$$

and the fact that Λ is a J-spectral factor for W, we have that

$$(G+K)^{\sim}(G+K)-\sigma^2 I_m = X_2^{-\sim}(Q^{\sim}Q-I_m)X_2^{-1}.$$

Thus $\|G+K\|_\infty < \sigma$, so, any K of the form $K = X_1X_2^{-1}$, where X_1 and X_2 are given by (4.13), is a solution for the sub-optimal Nehari extension problem. ∎

Remark 4.12 *The matrix-valued function $W(s)$, defined in (4.12) admits a J-spectral factorization. We can construct a J-spectral factor using the procedure described in the Algorithm 3.23. In order to solve the J-spectral factorization we need to solve two equations involving projection operators. For systems which have a state-space representation, an alternative to compute a J-spectral factor would be to solve two Riccati equations, which, for infinite-dimensional systems, it is not an easy task.*

Before we state a corollary of the Theorem 4.10 we define the Schmidt pairs of a compact operator.

Definition 4.13 *Let $T \in \mathcal{L}(Z_1, Z_2)$ be a compact operator, where Z_1 and Z_2 are two Hilbert spaces, and $\sigma_i \geq 0$ be the square roots of the eigenvalues of T^*T. The pairs (ψ_i, φ_i), where ψ_i and φ_i are the eigenvectors of T^*T and TT^*, respectively, coresponding to σ_i^2, and satisfying $T\psi_i = \sigma_i\varphi_i$ are called the* Schmidt pairs *of T.*

Corollary 4.14 *Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ be an equalizing vector for the matrix-valued function $W(s)$ defined in (4.12). The following assertions hold:*

1. u has the following representation

$$u = \begin{bmatrix} -\Pi_+ G u_2 \\ u_2 \end{bmatrix}. \tag{4.24}$$

2. u_2 is an eigenvector for the compact nonnegative operator $H_G^*H_G$ corresponding to the eigenvalue σ^2. Moreover, u_2 can be chosen to have norm one.

3. If $v = Wu \in H_2^\perp$, then

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} H_G u_2 \\ \Pi_- G^{\sim} H_G u_2 \end{bmatrix}. \tag{4.25}$$

4. $(u_2, \frac{v_1}{\sigma})$ is a Schmidt pair corresponding to σ, a nonzero singular value of the Hankel operator with symbol G.

Proof: *1.* Using (4.19), we see that u has the representation (4.24).

2. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ be an equalizing vector for the matrix-valued function $W(s)$, defined in (4.12). From (4.20), we see that u_2 is an eigenvector for $H_G^*H_G$

corresponding to the eigenvalue σ^2, and that, without loss of generality, u_2 can be choosen to have norm one.

3. We have that

$$v = Wu = \begin{bmatrix} I_k & G \\ G^{\sim} & G^{\sim}G - \sigma^2 I_m \end{bmatrix} \begin{bmatrix} -\Pi_+ G u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} H_G u_2 \\ \Pi_- G^{\sim} H_G u_2 \end{bmatrix}.$$

4. From (4.25) we see that $v_1 = H_G u_2$, so

$$H_G u_2 = \sigma \frac{v_1}{\sigma}$$

and using 2.

$$H_G^*(\frac{v_1}{\sigma}) = H_G^*(\frac{H_G u_2}{\sigma}) = \frac{1}{\sigma} H_G^* H_G u_2 = \sigma u_2.$$

■

Corollary 4.15 *If (ϕ, ψ) is the Schmidt pair of the Hankel operator with symbol G corresponding to a nonzero singular value σ, then*

$$u = \begin{bmatrix} -\Pi_+ G\phi \\ \phi \end{bmatrix}$$

is an equalizing vector for the matrix-valued function $W(s)$ defined in (4.12), and

$$Wu = \sigma \begin{bmatrix} \psi \\ \Pi_- G^{\sim}\psi \end{bmatrix}. \tag{4.26}$$

Proof: Let (ϕ, ψ) be the Schmidt pair of the Hankel operator with symbol G corresponding to a nonzero singular value σ. We have the following sequence of equalities:

$$\begin{aligned} Wu &= \begin{bmatrix} I_k & G \\ G^{\sim} & G^{\sim}G - \sigma^2 I_m \end{bmatrix} \begin{bmatrix} -\Pi_+ G\phi \\ \phi \end{bmatrix} \\ &= \begin{bmatrix} -\Pi_+ G\phi + G\phi \\ -G^{\sim}\Pi_+ G\phi + G^{\sim}G\phi - \sigma^2\phi \end{bmatrix} \\ &= \begin{bmatrix} \Pi_- G\phi \\ G^{\sim}\Pi_- G\phi - \sigma^2\phi \end{bmatrix} \\ &= \begin{bmatrix} \Pi_- G\phi \\ \Pi_- G^{\sim}\Pi_- G\phi + (\Pi_+ G^{\sim}\Pi_- G\phi - \sigma^2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_- G\phi \\ \Pi_- G^{\sim}\Pi_- G\phi + (H_G^* H_G\phi - \sigma^2\phi) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_- G\phi \\ \Pi_- G^{\sim}\Pi_- G\phi \end{bmatrix} \\ &= \begin{bmatrix} H_G\phi \\ \Pi_- G^{\sim} H_G\phi \end{bmatrix} \in H_2^{\perp}. \end{aligned}$$

This shows that u is an equalizing vector for the matrix-valued function $W(s)$ defined in (4.12). Since (ϕ, ψ) is the Schmidt pair of the Hankel operator with symbol G corresponding to the nonzero singular value σ, we have that $H_G\phi = \sigma\psi$. So, the relation (4.26) is satisfied. ∎

4.3 The Nehari extension problem and equalizing vectors

The problem of finding $K \in H_\infty^{k\times m}$ that achieve the minimum distance in

$$\inf_{K\in H_\infty^{k\times m}} \|G+K\|_\infty = \|H_G\|$$

is called the *Nehari extension problem.* The following theorems give connections between the equalizing vectors and the solutions of the Nehari extension problem.

Theorem 4.16 *Suppose that $G \in \hat{\mathcal{W}}^{k\times m}$. Then any $K_0 \in H_\infty^{k\times m}$ solving the Nehari extension problem, that is, satisfying*

$$\|G+K_0\|_\infty = \|H_G\| \tag{4.27}$$

also satisfies

$$(G+K_0)u_2 = H_G u_2, \tag{4.28}$$

where u_2 is an eigenvector for the compact nonnegative operator $H_G^*H_G$ corresponding to the largest eigenvalue $\|H_G\|^2$. Moreover, $G+K_0$ has constant modulus almost everywhere on the imaginary axis.

Proof: We have that the Hankel operator with symbol G is a compact operator (see [23], Lemma 8.1.7), and the equality

$$\|H_G u_2\|_{H_2^\perp} = \|H_G\|\|u_2\|_{H_2} \tag{4.29}$$

holds for u_2 an eigenvector for the compact nonnegative operator $H_G^*H_G$ corresponding to the largest eigenvalue $\|H_G\|^2$ (see [23], Lemma 8.1.12). The rest of the proof follows from Theorem 8.1.11 in [23]. ∎

The following theorem provides a connection between the equalizing vectors and solutions of the Nehari extension problem.

Theorem 4.17 *Let $\sigma = \|H_G\|$. Suppose that $G \in \hat{\mathcal{W}}^{k\times m}$ is a given matrix-valued function and that $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ is an equalizing vector for the matrix-valued function $W(s)$, defined in (4.12). If there exists a solution K_0 of the Nehari extension problem, then on the imaginary axis it satisfies*

$$K_0 u_2 = u_1. \tag{4.30}$$

Proof: Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ be an equalizing vector for the matrix-valued function $W(s)$, defined in (4.12). By Corollary 4.14.2. we know that u_2 is an eigenvector of $H_G^* H_G$ corresponding to the eigenvalue $\|H_G\|^2$. If K_0 is a solution for the Nehari extension problem, then by Theorem 4.16 it must satisfy

$$(G + K_0)u_2 = H_G u_2,$$

which is equivalent to

$$\begin{aligned} K_0 u_2 &= -Gu_2 + \Pi_- G u_2 \\ &= -\Pi_+ G u_2 \\ &= u_1 \qquad\qquad \text{from (4.24)} \end{aligned}$$

So, the equality (5.30) holds. ■

Remark 4.18 *From the relation (5.30) one can see that the equalizing vector is fixing the solution of the Nehari extension problem in the direction of the eigenvector corresponding to the largest singular value of the Hankel operator with symbol G.*

If the symbol is a scalar function, an equalizing vector can be used to prove the uniqueness of the solution for the Nehari extension problem.

Corollary 4.19 *Consider the scalar transfer function $g \in \hat{\mathcal{W}}$ and let $\sigma = \|H_g\|$. Suppose that $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ is an equalizing vector for the matrix-valued function $W(s)$ defined in (4.12), corresponding to σ. The solution k_0 of the Nehari extension problem is unique and on the imaginary axis it is given by*

$$k_0 = \frac{u_1}{u_2}. \tag{4.31}$$

Proof: Since $u_2 \in H_2$, it is zero, at most, on a set of measure zero (see [23], Lemma A.6.20) of the imaginary axis. This means that we can divide the equality (5.30) by u_2 and obtain (4.31). ■

Remark 4.20 *The Nehari extension problem corresponding to every $G \in \hat{\mathcal{A}}^{\sim,k\times m}$ has a solution in H_∞. Let us denote it by K. If $(\gamma_n)_{n\in\mathbb{N}}$ is a decreasing sequence with limit $\|H_G\|$ and $(K_n)_{n\in\mathbb{N}}$ is a corresponding sequence of solutions (can be chosen rational) for the sub-optimal Nehari extension problems, then there exists a subsequence $K_{\alpha(n)}$ such that*

$$\lim_{n\to\infty} \langle K_{\alpha(n)} f(s), g(s)\rangle_{L_2} = \langle K f(s), g(s)\rangle_{L_2}$$

for every $f \in L_2^m$ and every $g \in L_2^k$.

A proof for the results stated in the previous remark is similar to that in Curtain and Zwart [23](see Theorem 8.3.9).

Remark 4.21 *For the scalar case, the previous corollary gives a solution for the Nehari extension problem, providing that we have an equalizing vector. From Theorem 4.16 we have that* $g + k_0$ *has constant modulus almost everywhere on the imaginary axis. This means that once we have an equalizing vector, we find a* k_0 *which "equalizes"* g *over the imaginary axis (complete* g *to a function of constant modulus almost everywere on the imaginary axis).*

4.4 The Nehari problem for Pritchard-Salamon systems

In this section we consider the Nehari problem for exponentially stable Pritchard-Salamon systems. In this way we provide the connection between the frequency-domain result obtained in the first section and the state-space representation (of PS systems).

Before we can state the main resut we need to recall some of the state-space concepts.

Definition 4.22 *Let* $(T(\cdot), B, C, D)$ *be an exponentially stable smooth Pritchard-Salamon system. Its* controllability operator $\mathcal{B} \in \mathcal{L}(L_2^m(0,\infty), W)$ *is defined by*

$$\mathcal{B}u = \int_0^\infty T(s)Bu(s)ds$$

and its observability operator $\mathcal{C} \in \mathcal{L}(V, L_2^n(0,\infty))$ *is defined by*

$$\mathcal{C}v = CT(t)v \text{ for } v \in W.$$

Its controllability gramian $L_B \in \mathcal{L}(W^*, W)$ *and its* observability gramian $L_C \in \mathcal{L}(V, V^*)$ *are defined by*

$$\begin{aligned} L_B &= \mathcal{B}\mathcal{B}^* = \int_0^\infty T(t)BB^*T^*(t)dt, \\ L_C &= \mathcal{C}^*\mathcal{C} = \int_0^\infty T^*(t)C^*CT(t)dt, \end{aligned}$$

respectively.

The sub-optimal Nehari problem for exponentially stable, smooth Pritchard-Salamon systems with infinite-dimensional input and output spaces was solved by Curtain and Zwart in [22]. The solution proposed in their paper uses the particular realization available for Pritchard-Salamon systems. Using the fact that the transfer function of an exponentially stable, smooth Pritchard-Salamon system is in the Wiener algebra on the right half-plane (see Lemma 2.48) our Theorem 4.7 holds also for this class of systems. However, only for the case when the input and output spaces are finite dimensional. This is since the Hankel operator associated with an exponentially stable Pritchard-Salamon system with finite-dimensional input and output spaces is compact. In general, this is not the case, and it is illustrated with a simple example in Curtain and Zwart [22], Example 2.9.

The main advantage of the use of state space realization for exponentially stable, smooth Pritchard-Salamon systems is the fact that we can provide an explicit representation for the J-spectral factor.

Lemma 4.23 *Suppose that $\Sigma(A,B,C,D)$ is an exponentially stable, smooth Pritchard-Salamon system, with $G(s)^{\sim}$ its transfer function and $\|H_G\| < \sigma$. Let us define $\Lambda(s)^{\sim}$ by*

$$\Lambda(s)^{\sim} = \begin{bmatrix} I_n & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2} \begin{bmatrix} -CL_B \\ \sigma B^* \end{bmatrix} N^*(sI + A^*)^{-1} \begin{bmatrix} C^* & L_C B \end{bmatrix} \tag{4.32}$$

where $N = (I - \sigma^{-2} L_B L_C)^{-1}$.
Then the following properties are satisfied:

1. *$\Lambda(s)^{\sim}$ is invertible and analytic in the left-half plane;*
2. *$\Lambda(s)^{-\sim}$ is given by*
$$\Lambda(s)^{-\sim} = \begin{bmatrix} I_n & 0 \\ 0 & \sigma^{-1} I_m \end{bmatrix} - \sigma^{-2} \begin{bmatrix} -CL_B \\ \sigma B^* \end{bmatrix} N^*(sI+A^*)^{-1} \begin{bmatrix} C^* & \sigma^{-1} L_C B \end{bmatrix}; \tag{4.33}$$
3. *The matrix-valued function*
$$W(s) = \begin{bmatrix} I_n & 0 \\ G^{\sim}(s) & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_n & G(s) \\ 0 & I_m \end{bmatrix} \tag{4.34}$$
has the J-spectral factorization
$$W(s) = \Lambda^{\sim}(s) \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(s);$$
4. *$\Lambda_{11}(s)^{\sim}$ is invertible;*
5. *$\Lambda(s)$, $\Lambda^{-1}(s)$ and $\Lambda_{11}(s)^{-1}$ are transfer operators of exponentially stable smooth Pritchard-Salamon systems, and consequently in $\hat{\mathcal{A}}$.*

Proof: For a proof of this result we refer to Curtain and Zwart [22], Lemma 2.11. ■

For exponentially stable, smooth Pritchard-Salamon systems we have two alternatives of constructing the J-spectral factor: one can use the state-space representation as in Lemma 4.23 or one can use a frequency-domain approach as presented in Algorithm 3.23.

Combining Theorem 4.7 with Lemma 4.23 we obtain, for exponentially stable, smooth Pritchard-Salamon systems with finite-dimensional input and output spaces the following result.

Theorem 4.24 *Suppose that $\Sigma(A,B,C,D)$ is a given exponentially stable, smooth Pritchard-Salamon system with finite rank inputs and outputs spaces, $G(s)^{\sim}$ its transfer function, and σ a positive real number. The following statements are equivalent:*

1. $\|H_G\| < \sigma$.
2. *There exists $K(s) \in \hat{\mathcal{A}}^{k\times m}$ such that*
$$\|G+K\|_\infty < \sigma.$$
3. *The matrix-valued function $\Lambda(s) \in \hat{\mathcal{A}}^{(k\times m)\times(k\times m)}$ given by*
$$\Lambda(s)^{\sim} = \begin{bmatrix} I_n & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2}\begin{bmatrix} -CL_B \\ \sigma B^* \end{bmatrix} N^*(sI+A^*)^{-1}\begin{bmatrix} C^* & L_C B \end{bmatrix}$$
is a J-spectral factor for
$$W(s) = \begin{bmatrix} I_k & 0 \\ G^{\sim}(s) & I_m \end{bmatrix}\begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix}\begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}$$
with $\Lambda_{11}^{-1}(s) \in \hat{\mathcal{A}}^{k\times k}$.

Furthermore, if this conditions hold, all solutions for the sub-optimal Nehari extension problem are parametrized by
$$K(s) = X_1(s)X_2(s)^{-1},$$
where
$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1}\begin{bmatrix} Q(s) \\ I_m \end{bmatrix}, \tag{4.38}$$
with $Q(s) \in \hat{\mathcal{A}}^{k\times m}$, $\|Q\|_\infty < 1$.

Chapter 5

The Hankel norm approximation problem

Introduction

The sub-optimal Hankel norm approximation problem has been studied extensively in the literature (see for example, Adamjan et al. [1], Ball and Ran [7], Glover [28], Ran [59], Glover et al. [29], Curtain and Ran [19], Sasane [63], Sasane and Curtain [65]). The new contribution of this chapter is to present an elementary derivation of the reduction of the sub-optimal Hankel norm approximation problem to a J-spectral factorization problem. We do this for the Wiener class of matrix-valued functions. The solution of this J-spectral factorization problem can then be obtained by solving two equations (see Algorithm 3.23). Moreover an explicit parameterization of all solutions to the sub-optimal Hankel norm approximation problem is provided.

Although not stated explicitly in their paper, we believe that the paper [5] of Ball and Helton is the first which shows the connection between the sub-optimal Hankel norm approximation problem and a J-spectral factorization problem. Various corollaries of this abstract paper have been derived in Ball and Ran [7] and Curtain and Ran [19], but there is a gap between the abstract theory in [5] and the elementary looking corollaries. This motivated the search for an elementary self-contained proof in many papers (see Curtain and Ichikawa [16], Curtain and Oostveen [18], Curtain and Zwart [22], Sasane and Curtain [65], [67] and Iftime and Zwart [40]).

The results presented in this chapter refines and/or generalizes the preceding lemmas in Curtain and Ichikawa [16], Curtain and Zwart [22], Sasane and Curtain [65], [67]. All the proofs are based on frequency domain techniques. In the last section we provide formulas for a spectral factor in terms of the state space parameters A, B and C for the Pritchard-Salamon class of infinite-dimensional systems.

5.1 Preliminaries

Before we formulate the sub-optimal Hankel norm approximation problem and state the main results we need to introduce some notations and prove preliminary results.

We denote by $\hat{\mathcal{A}}_l^{n\times m}$ the set of complex $n \times m$ matrix-valued functions K of a complex variable with a decomposition

$$K = G + F, \tag{5.1}$$

where $F \in \hat{\mathcal{A}}^{n\times m}$ and G is a proper matrix-valued rational transfer function of a system with MacMillan degree *at most equal* to l, and with all its poles in the open right half-plane. The subset of $\hat{\mathcal{A}}_l^{n\times m}$ containing all matrix-valued transfer functions with the decomposition (5.1) such that G is the matrix-valued rational transfer function of a system of MacMillan degree equal to l, will be denoted by $\hat{\mathcal{A}}_{[l]}^{n\times m}$.

The *MacMillan degree* of a proper rational matrix-valued function G is defined to be the minimal state-space dimension of all possible (A, B, C, D)'s that realize G, i.e.,

$$\text{MacMillan degree} = \min_{\substack{A\in\mathbb{C}^{n\times n}, B\in\mathbb{C}^{n\times m},\\ C\in\mathbb{C}^{p\times n}, D\in\mathbb{C}^{p\times m}}} \{n \in \mathbb{N};\ G(s) = C(sI - A)^{-1}B + D\}.$$

If the transfer function $G(s)$ is a scalar rational function, then its MacMillan degree equals its number of unstable poles, see Kailath [43]. Using this relation the following lemma easily follows.

Lemma 5.1 *If $f \in \hat{\mathcal{A}}$ and $g \in \hat{\mathcal{A}}_\infty$ are such that g has at most l unstable zeros (all in the open right half-plane), then $\frac{f}{g} \in \hat{\mathcal{A}}_l$.*

Proof: From the definition of $\hat{\mathcal{B}}_0$ we see that $\frac{f}{g} \in \hat{\mathcal{B}}_0$. Using Lemma 2.29 we obtain that

$$\frac{f}{g} = \hat{a} + \hat{r},$$

where $\hat{a} \in \hat{\mathcal{A}}$ and $\hat{r}$ is a strictly proper, rational transfer function with all its poles in the open right half-plane. The poles of $\hat{r}$ must be the zeros of g, so we can conclude that $\frac{f}{g} \in \hat{A}_l$. ■

An alternative proof of the previous lemma can be obtained in an analogous manner as in Sasane [64] (Lemma 2.5.1, page 53).

We give conditions under which the inverse of a matrix-valued function $F \in \hat{\mathcal{A}}^{k\times k}$ exists and belongs to $\hat{\mathcal{A}}_l^{k\times k}$.

Lemma 5.2 *Let $F \in \hat{\mathcal{A}}^{k\times k}$. If*

1. *$F(s)$ is invertible for every $s \in \mathbb{C}_0$,*

2. $\det F \in \hat{\mathcal{A}}_\infty$,

then $F^{-1} \in \hat{\mathcal{A}}_l^{k\times k}$*, where* $l \in \mathbb{N}$ *is the number of unstable zeros of* $\det F$*, counted with respect to their multiplicities.*

Proof: The inverse of the square matrix $F(s)$ is given by

$$F(s)^{-1} = \frac{1}{\det F(s)} \operatorname{adj}(F(s)),$$

where the components of $\operatorname{adj}(F)$ are sums and products of components of F. Since $F \in \hat{\mathcal{A}}^{k\times k}$, we have that $\operatorname{adj}(F) \in \hat{\mathcal{A}}^{k\times k}$. From the assumption $\det F \in \hat{\mathcal{A}}_\infty$ and Lemma 5.1 we have that $F^{-1} \in \hat{\mathcal{A}}_l^{k\times k}$, where $l \in \mathbb{N}$ is the number of unstable poles of $\det F$, counted with respect to their multiplicities. ■

The following lemma gives a characterization of the set $\hat{\mathcal{B}}_0$ (see Definition 2.27).

Lemma 5.3 *The following equality hods*

$$\hat{\mathcal{B}}_0^{n\times m} = \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_{[l]}^{n\times m}. \tag{5.2}$$

Proof: Let us consider an element $K \in \hat{\mathcal{B}}_0^{n\times m}$. From the definition of $\hat{\mathcal{B}}_0^{n\times m}$ we have that K is a matrix-valued function with entries in $\hat{\mathcal{B}}_0$. As a consequence of Lemma 2.29, K has the representation

$$K = G + F,$$

where $F \in \hat{\mathcal{A}}^{n\times m}$ and G has all its entries proper, rational transfer function with all there poles in the open right half-plane. G is the transfer function of a finite-dimensional system. Let us denote its MacMillan degree with $l \in \mathbb{N}$. This means that for this $l \in \mathbb{N}$ we have that $K \in \hat{\mathcal{A}}_{[l]}^{n\times m}$. This proves the inclusion

$$\hat{\mathcal{B}}_0^{n\times m} \subset \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_{[l]}^{n\times m}.$$

We prove now the other inclusion. Let us consider $K \in \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_{[l]}^{n\times m}$. Then there exist an $l \in \mathbb{N}$ such that $K \in \hat{\mathcal{A}}_{[l]}^{n\times m}$. This means that $K = G + F$, where G is a proper rational matrix-valued transfer function of a system of MacMillan degree equal to l, with all its poles in the open right half-plane, and $F \in \hat{\mathcal{A}}^{n\times m}$. Using once more Lemma 2.29, we obtain that $K \in \hat{\mathcal{B}}_0^{n\times m}$. This completes the proof. ■

Remark 5.4 *By definition of* $\hat{\mathcal{A}}_{[l]}^{n\times m}$ *and* $\hat{\mathcal{A}}_l^{n\times m}$ *we see that*

$$\bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_l^{n\times m} = \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_{[l]}^{n\times m},$$

so

$$\hat{\mathcal{B}}_0^{n\times m} = \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_{[l]}^{n\times m} = \bigcup_{l\in\mathbb{N}} \hat{\mathcal{A}}_l^{n\times m}.$$

Combining the results stated in Lemma 2.37 and Lemma 5.3 we conclude that the elements in $\hat{\mathcal{A}}_{[l]}^{n\times m}$ have a right(left)-coprime factorization.

Lemma 5.5 *For any $K \in \hat{\mathcal{A}}_{[l]}^{n\times m}$ there exists a right-coprime factorization $K = NM^{-1}$ over $\hat{\mathcal{A}}$, where M is rational and $\det(M) \in \hat{\mathcal{A}}_\infty$. Moreover $\det(M)$ has exactly l poles and they are all contained in the open right half-plane. Any two right-coprime factorizations of K over $\hat{\mathcal{A}}$ are unique up to a multiplication to the right by a matrix-valued function inverible over $\hat{\mathcal{A}}^{m\times m}$. (A similar statement holds for left-coprime factorization.)*

Before we state another technical result, we introduce a useful notation. For any fixed $\zeta \in \mathbb{R}$ we denote by

$$\|K(\zeta + \cdot)\|_{L_\infty} := \sup_{s\in\mathbb{C}_0} \|K(\zeta + s)\| \tag{5.3}$$

the L_∞ norm on a line paralel with the imaginary axis for a matrix-valued functions $K \in \hat{\mathcal{A}}_l$. For $\zeta = 0$ the norm defined above is exactly the L_∞-norm on the imaginary axis.

Lemma 5.6 *If $K \in \hat{\mathcal{A}}_l$ for some $l \in \mathbb{N}$, then given any $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 \leq \zeta \leq \delta$, we have*

$$\|K(\zeta + \cdot)\|_{L_\infty} \leq \|K(\cdot)\|_{L_\infty} + \varepsilon, \tag{5.4}$$

where $\|\cdot\|_{L_\infty}$ is defined in (5.3) .

Proof: We use the compactification of the set $\overline{\mathbb{C}_+}\bigcup\{\infty\}$ and hence treat it as a compact set (the compactification point ∞ was defined in the first section of Chapter 3). Let $K \in \hat{\mathcal{A}}_l$ for some $l \in \mathbb{N}$, and consider any real $\varepsilon > 0$. K is continuous on the compact set $\mathbb{C}_+\bigcup\{\infty\}$. So, there exists a $\delta > 0$ such that

$$|K(\zeta + s)| \leq |K(s)| + \varepsilon,$$

for all $\zeta \in [0, \delta]$ and $s \in \mathbb{C}_0\bigcup\{\infty\}$. Taking the supremum over $s \in \mathbb{C}_0\bigcup\{\infty\}$ first in the right-hand side then in the left-hand side, we obtain the inequality (5.4). ■

Lemma 5.7 *If $K \in \hat{\mathcal{A}}_l^{n\times m}$, $K_1 \in \hat{\mathcal{A}}^{n_*\times n}$, and $K_2 \in \hat{\mathcal{A}}^{m\times m_*}$, then*

$$K_1 K K_2 \in \hat{\mathcal{A}}_l^{n_*\times m_*}.$$

Proof: Let us consider $K \in \hat{\mathcal{A}}_l^{n\times m}$, $K_1 \in \hat{\mathcal{A}}^{n_*\times n}$, and $K_2 \in \hat{\mathcal{A}}^{m\times m_*}$. From Lemma 2.37 we have that K has a coprime factorization $K = NM^{-1}$ over $\hat{\mathcal{A}}$ and $\det M \in \hat{\mathcal{A}}_\infty$. Thus

$$K_1 K K_2 = \frac{1}{\det(M)} K_1 N \operatorname{adj}(M) K_2,$$

where $K_1, K_2, \mathrm{adj}(M) \in \hat{\mathcal{A}}$. It is easy to see that $K_1 K K_2 \in \hat{\mathcal{B}}_0$. The poles of $K_1 K K_2$ will be at most the zeros of $\det(M)$. Using now Lemma 5.3 and the fact that

$$\hat{\mathcal{A}}_k \subset \hat{\mathcal{A}}_l \text{ for } k \leq l,$$

we obtain that $K_1 K K_2 \in \hat{\mathcal{A}}_l$. ■

An analytic function has only isolated zeros (see Lemma A.1.4 in Curtain and Zwart [23]). We recall the concept of the Nyquist index of an analytic function. (This concept is generaly defined for *meromorphic functions*, which are functions that can be writen as a ratio of two analytic functions.)

Definition 5.8 *Let g be an analytic function on $\mathbb{C}_{+,-\epsilon}$ for some $\epsilon > 0$ and suppose that g has a nonzero limit at ∞ over $\overline{\mathbb{C}_+}$ and g has no zeros on the imaginary axis. We define the number of times the plot of $g(s)$ encircles the origin in a couterclockwise sense as s decreases from $j\infty$ to $-j\infty$ over the imaginary axis to be the* Nyquist index *of g.*

Since an analytic function on $\mathbb{C}_{+,-\epsilon}$ has no poles in $\overline{\mathbb{C}_+}$, its Nyquist index is equal to the number of zeros in $\overline{\mathbb{C}_+}$.

The following lemma is a particular case of Lemma A.1.18 in Curtain and Zwart [23].

Lemma 5.9 *Let g_1 and g_2 be analytic functions on an open set containing $\overline{\mathbb{C}_+}$ with nonzero limits at infinity $g_1(\infty)$ and $g_2(\infty)$, and no zeros on the imaginary axis. If there exists a continuous function $\phi : [0,1] \times \mathbb{C}_0 \to \mathbb{C}$ such that*

1. *$\phi(0,s) = g_1(s)$, $\phi(1,s) = g_2(s)$;*
2. *$\phi(\alpha,s)$ and $\phi(\infty,s)$ are nonzero for all $\alpha \in [0,1]$ and $s \in \mathbb{C}_0$,*

then the Nyquist indices of g_1 and g_2 are the same.

5.2 The sub-optimal Hankel norm approximation problem

The Hankel operator with symbol $G \in L_\infty^{k\times m}$, H_G and its adjoint H_G^* were defined in (4.9) and (4.10), respectively. We recall that the Hankel operator with symbol $G \in \hat{\mathcal{W}}^{k\times m}$ is compact, the Hankel singular values σ_i are countable (we count them with respect to their multiplicities), $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$.

The sub-optimal Hankel norm approximation problem for the Wiener class is the following: given $G \in \hat{\mathcal{W}}^{k\times m}$ and σ satisfying $\sigma_{l+1} < \sigma < \sigma_l$, find $K \in \hat{\mathcal{A}}_l^{k\times m}$ such that

$$\|G + K\|_{L_\infty} \leq \sigma.$$

Adamjan, Arov and Krein [1] (for the scalar case) and Sasane [64] (for the matrix-valued case) proved that

$$\inf_{K \in \hat{\mathcal{A}}_l^{k\times m}} \|G + K\|_{L_\infty} = \sigma_{l+1}. \tag{5.5}$$

This result leads us to the following remark.

Remark 5.10 *Any solution to the sub-optimal Hankel norm approximation problem has an unstable rational part of MacMillan degree exactly l.*

The following theorem is a consequence of a slightly more general result proved in Sasane and Curtain [65]. They give sufficient conditions for the sub-optimal Hankel norm approximation problem to have a solution.

Theorem 5.11 *Let us make the following assumptions:*

S1. $G \in \hat{\mathcal{W}}^{k\times m}$ *such that* $G^{\sim} \in \hat{\mathcal{A}}^{m\times k}$.

S2. $\sigma_{l+1} < \sigma < \sigma_l$,

S3. There exists a $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *such that*

$$\begin{bmatrix} I_k & 0 \\ G(s)^* & I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix} = \Lambda(s)^* \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(s), \tag{5.6}$$

for all $s \in \mathbb{C}_0$,

S4. Λ *is invertible as an element of* $\hat{\mathcal{A}}^{(k+m)\times(k+m)}$, *that is, there exists a* $V \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *such that* $\Lambda(s)V(s) = I_{k+m}$ *for all* $s \in \overline{\mathbb{C}_+}$,

S5. At infinity we have that

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} \Lambda(s) = \begin{bmatrix} I_k & 0 \\ 0 & \sigma I_m \end{bmatrix},$$

S6. $\Lambda_{11}^{-1} \in \hat{\mathcal{A}}_l^{k\times k}$.

Then $K \in \hat{\mathcal{A}}_l^{k\times m}$ *and* $\|G+K\|_{L_\infty} \le \sigma$ *if and only if* $K = X_1 X_2^{-1}$, *where*

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix}$$

for some $Q \in \hat{\mathcal{A}}^{k\times m}$ *satisfying* $\|Q\|_{H_\infty} \le 1$.

In this chapter we show that the assumptions S3-S5 are not necessary for solving the sub-optimal Hankel norm approximation problem for the Wiener class. The main result is the following:

Theorem 5.12 *Consider* $G^{\sim} \in \hat{\mathcal{A}}_0^{m\times k}$ *and let* σ *be such that* $\sigma \ne \sigma_p$ *for* $p \in \mathbb{N}$. *Then, there exists a* $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *such that*

a. the following quality holds

$$\begin{bmatrix} I_k & 0 \\ G(s)^* & I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix} = \Lambda(s)^* \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(s),$$

for all $s \in \mathbb{C}_0$,

b. Λ *is invertible as an element of* $\hat{\mathcal{A}}^{(k+m)\times(k+m)}$, *that is, there exists a* $V \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *such that* $\Lambda(s)V(s) = I_{k+m}$ *for all* $s \in \overline{\mathbb{C}_+}$,

c. At infinity we have that

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} \Lambda(s) = \begin{bmatrix} I_k & 0 \\ 0 & \sigma I_m \end{bmatrix},$$

and the following two statements are equivalent:

1. *There exists a* $K \in \hat{\mathcal{A}}_l^{k\times m}$ *such that* $\|G+K\|_{L_\infty} \leq \sigma$.

2. $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *satisfies S6 from Theorem 5.11.*

Furthermore, all solutions to the sub-optimal Hankel norm approximation problem are given by

$$K(s) = X_1(s)X_2(s)^{-1},$$

where

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix}. \tag{5.11}$$

for some $Q \in \hat{\mathcal{A}}^{k\times m}$ *satisfying* $\|Q\|_{H_\infty} \leq 1$.

We remark that a., b., c. from Theorem 5.12 are nothing other that the assumptions S3., S4., S5. from Theorem 5.11.

Lemma 5.13 *Consider* $G^{\sim} \in \hat{\mathcal{A}}^{m\times k}$ *and let* σ *be such that* $\sigma \neq \sigma_p$ *for* $p \in \mathbb{N}$. *Then the assumptions S3-S5 are satisfied.*

Proof: This lemma is a direct consequence of Theorem 4.10. ■

Before we proceed with the proof of Theorem 5.12 we need to add some more preliminary results.

Lemma 5.14 *Consider* $G^{\sim} \in \hat{\mathcal{A}}_0^{m\times k}$ *and let* σ *be such that* $\sigma \neq \sigma_p$ *for* $p \in \mathbb{N}$, *and* $V \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ *be the inverse of* Λ *over* $\hat{\mathcal{A}}$. *Then the following statements hold*

1. *The limit of* V *at infinity is*

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} V(s) = \begin{bmatrix} I_k & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix}. \tag{5.12}$$

2. $V_{22}(s)^{-1}$ *exists for all* $s \in \mathbb{C}_0$ *and*

$$\|V_{22}^{-1}V_{21}\|_{L_\infty} < 1. \tag{5.13}$$

Proof: By Lemma 5.13, we have that S3-S5 are satisfied.

1. From S5 we have that
$$V(s) - \begin{bmatrix} I_k & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} = V(s)\left(\begin{bmatrix} I_k & 0 \\ 0 & \sigma I_m \end{bmatrix} - \Lambda(s) \right) \begin{bmatrix} I_k & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix}.$$
Using the fact that $V \in \hat{\mathcal{A}}^{(k+m)\times(k+m)} \subset H_\infty$, it follows that the limit of V at infinity is the one given in (5.12).

2. The matrix-valued function $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ satisfies (5.6), and so taking inverses, we obtain
$$V(s) \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} V(s)^* =$$
$$\begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} I_k & 0 \\ 0 & -\frac{1}{\sigma^2} I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ G(s)^* & I_m \end{bmatrix}^{-1}.$$
for all $s \in \mathbb{C}_0$. Considering the $(2,2)-$block of the above matrix-functions equality, yields
$$V_{21}(s)V_{21}(s)^* - V_{22}(s)V_{22}(s)^* = -\frac{1}{\sigma^2} I_m, \tag{5.14}$$
for all $s \in \mathbb{C}_0$. Thus for $u \in \mathbb{C}^m$ we have that
$$\|V_{22}(s)^* u\|^2 = \|V_{21}(s)^* u\|^2 + \frac{1}{\sigma^2}\|u\|^2,$$
for all $s \in \mathbb{C}_0$. So, if $V_{22}(s)^* u = 0$ for all $s \in \mathbb{C}_0$, then $u = 0$. Hence it follows that $V_{22}(s)^*$ is invertible for all $s \in \mathbb{C}_0$, or equivalently, $V_{22}(s)$ is invertible for all $s \in \mathbb{C}_0$.
From (5.14), we have that
$$\left\|V_{21}(s)^* V_{22}(s)^{-*} u\right\|^2 - \|u\|^2 = -\frac{1}{\sigma^2} \left\|V_{22}(s)^{-*} u\right\|^2.$$
for all $s \in \mathbb{C}_0$. Let $M > 0$ be such that $\|V_{22}(s)^*\| \leq M$ for all $s \in \mathbb{C}_0$. We obtain that
$$\|u\|^2 \leq \|V_{22}(s)^*\|^2 \left\|V_{22}(s)^{-*} u\right\|^2 \leq M^2 \left\|V_{22}(s)^{-*} u\right\|^2.$$
Thus, since $\|R\| = \|R^*\|$, the following inequality holds
$$\left\|V_{22}(s)^{-1} V_{21}(s)\right\|^2 = \left\|V_{21}(s)^* V_{22}(s)^{-*}\right\|^2 \leq 1 - \frac{1}{\sigma^2 M^2}$$
for all $s \in \mathbb{C}_0$, and so we have $\left\|V_{22}^{-1} V_{21}\right\|_{L_\infty} < 1$ which is exactly the inequality (5.13).

So both items are proved. ■

The following lemma provides more information about the $(1,1)-$block of the J-spectral factor $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$.

Lemma 5.15 *Let $G^{\sim} \in \hat{\mathcal{A}}_0^{m\times k}$, and suppose that there exists a $K \in \hat{\mathcal{A}}_l^{k\times m}$ such that $\|G+K\|_{L_\infty} \leq \sigma$, where $\sigma \neq \sigma_p$ for $p \in \mathbb{N}$. Let $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ be a J-spectral factor (satisfying S3, S4 and S5), and $V = \Lambda^{-1}$. Then $V_{22}^{-1} \in \hat{\mathcal{A}}_l^{k\times k}$. Moreover $\Lambda_{11}^{-1} \in \hat{\mathcal{A}}_l^{k\times k}$, that is S6 holds .*

Proof: We will split the proof in four steps. First we prove that $V_{22}^{-1} \in \hat{\mathcal{A}}_{[l_*]}^{m\times m}$ for some $l_* \in \mathbb{N}$. In the second step we define

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} := \Lambda \begin{bmatrix} N \\ M \end{bmatrix},$$

where $K = NM^{-1}$ is a right-coprime factorization of K over $\hat{\mathcal{A}}$, and prove that U_2 is invertible over the imaginary axis and

$$\|U_1U_2^{-1}\|_{L_\infty} \leq 1.$$

Using the Nyquist index, in Step 3. we show that $l_* \leq l$. Finally, in the last step of the proof it is shown that S6 holds.

Step 1. We prove first that $V_{22}^{-1} \in \hat{\mathcal{A}}_{[l_*]}^{m\times m}$ for some $l_* \in \mathbb{N}$. From (5.12), we know that

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} V_{22}(s) = \frac{1}{\sigma} I_m.$$

We have also, from Lemma 5.14.2, that $V_{22}(s)$ is invertible for all $s \in \mathbb{C}_0$. Since $V_{22} \in \hat{\mathcal{A}}^{m\times m}$ we have that $\det(V_{22}) \in \hat{\mathcal{A}}$. Moreover, $\det(V_{22})$ has no zeros on the imaginary axis and a nonzero limit at infinity. This implies that $\det(V_{22}) \in \hat{\mathcal{A}}_\infty$. Thus applying Lemma 5.2 for $V_{22} \in \hat{\mathcal{A}}^{m\times m}$, we obtain that $V_{22}^{-1} \in \hat{\mathcal{A}}_{[l_*]}^{m\times m}$ for some $l_* \in \mathbb{N}$.

Step 2. Let $K \in \hat{\mathcal{A}}_l^{k\times m}$ satisfy $\|G+K\|_{L_\infty} \leq \sigma$ and let $K = NM^{-1}$ be a right-coprime factorization of K over $\hat{\mathcal{A}}$, where $N \in \hat{\mathcal{A}}^{k\times m}$ and $M \in \hat{\mathcal{A}}^{m\times m}$. Since $\hat{\mathcal{A}}_l \subset \hat{\mathcal{B}}_0$ (see Remark 5.4), we may take M to be rational, and $\det M \in \hat{\mathcal{A}}_\infty$ with at most $l \in \mathbb{N}$ zeros in the open right-half plane, none of them on the imaginary axis (see Lemma 5.5). Define

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} := \begin{bmatrix} \Lambda_{11}N + \Lambda_{12}M \\ \Lambda_{21}N + \Lambda_{22}M \end{bmatrix} = \Lambda \begin{bmatrix} N \\ M \end{bmatrix} = \Lambda \begin{bmatrix} K \\ I_m \end{bmatrix} M. \tag{5.15}$$

We prove that U_2 is invertible over the imaginary axis and $\|U_1U_2^{-1}\|_{L_\infty} \leq 1$. First we prove that $\ker(U_2(s)) = \{0\}$ for all $s \in \mathbb{C}_0$. From (5.15) we have that

$$\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} = \Lambda(s) \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} G(s)+K(s) \\ I_m \end{bmatrix} M(s), \tag{5.16}$$

for all $s \in \mathbb{C}_0$. Note that the following equality holds

$$U_1(s)^*U_1(s) - U_2(s)^*U_2(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}^* \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},$$

for all $s \in \mathbb{C}_0$. Multiplying the equality (5.6), to the left and to the right with apropriate matrices we have that

$$\begin{bmatrix} I_k & 0 \\ G(s)^* & I_m \end{bmatrix}^{-1} \Lambda(s)^* \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(s) \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix}^{-1} = \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix},$$

for all $s \in \mathbb{C}_0$. Using (5.16) and the above equality, we obtain that

$$U_1(s)^* U_1(s) - U_2(s)^* U_2(s) = M(s)^* \begin{bmatrix} G(s)+K(s) \\ I_m \end{bmatrix}^* \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} G(s)+K(s) \\ I_m \end{bmatrix} M(s) \quad (5.18)$$

on the imaginary axis. Hence for all $u \in \mathbb{C}^m$ and all $s \in \mathbb{C}_0$, we have from equation (5.18) that

$$\|U_1(s)u\|^2 - \|U_2(s)u\|^2 = \|(G(s)+K(s))M(s)u\|^2 - \sigma^2 \|M(s)u\|^2, \quad (5.19)$$

for $s \in \mathbb{C}_0$. Since $\|G(s)+K(s)\|_\infty \leq \sigma$ and $M(s)$ is invertible on the imaginary axis, we can conclude that $U_1(s)$ and $U_2(s)$ satisfy the following inequality

$$\|U_1(s)u\| \leq \|U_2(s)u\|. \quad (5.20)$$

for $s \in \mathbb{C}_0$. Multiplying to the left the equality (5.15) with V, the inverse of Λ, we obtain that

$$V \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} K \\ I_m \end{bmatrix} M,$$

and so

$$V_{21} U_1 + V_{22} U_2 = M, \quad (5.21)$$

which is equivalent to

$$V_{22}^{-1} V_{21} U_1 + U_2 = V_{22}^{-1} M \quad (5.22)$$

on the imaginary axis. We would like to prove that $\ker(U_2(s)) = \{0\}$ for all $s \in \mathbb{C}_0$. Suppose on the contrary that there exists $0 \neq u_0 \in \mathbb{C}^m$ and a $s_0 \in \mathbb{C}_0$ such that $U_2(s_0)u_0 = 0$. Then, from (5.13), (5.20) and (5.22), we obtain $V_{22}^{-1}(s_0)M(s_0)u_0 = 0$. Since $V_{22}^{-1}(s_0)M(s_0)$ is invertible, this implies that $u_0 = 0$, which is a contradiction. We have that $\ker(U_2(s)) = \{0\}$ for all $s \in \mathbb{C}_0$.
From (5.19), we deduce that

$$\left\|U_1(s)U_2(s)^{-1}y\right\|^2 \leq \|y\|^2,$$

for all $s \in \mathbb{C}_0$ and $y \in \mathbb{C}^m$. So, $U_1 U_2^{-1} \in L_\infty^{k\times m}$ and satisfies

$$\left\|U_1 U_2^{-1}\right\|_{L_\infty} \leq 1.$$

Step 3. We prove, using the Nyquist index, that $l_* \leq l$, where $l_* \in \mathbb{N}$ is the one from step 1. Consider

$$U_2 := \Lambda_{21} N + \Lambda_{22} M \in \hat{\mathcal{A}}^{m\times m}$$

as defined in (5.15). We know from S5 that Λ_{21} is strictly proper and Λ_{22} is proper with invertible limits at infinity in $\overline{\mathbb{C}_+}$. By construction, M is rational and $\det(M) \in \hat{\mathcal{A}}_\infty$. Thus we see that

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_+}}} U_2(s) \text{ exists and is invertible.} \tag{5.23}$$

We have proved (in step 2) that U_2 is invertible over the imaginary axis, so $\det(U_2)$ has no zeros on $\mathbb{C}_0$. We see that $\det(U_2) \in \hat{\mathcal{A}}_\infty$, and we can apply Lemma 5.2 to conclude that $U_2^{-1} \in \hat{\mathcal{A}}_{[q]}^{m\times m}$ for some $q \in \mathbb{N}$.

The zeros of $\det(V_{22})$, $\det(M)$ and $\det(U_2)$ are contained in some half-plane $\mathbb{C}_{+,\varepsilon}$, where $\varepsilon > 0$. Since from Lemma 5.14.2 we have that $\left\|V_{22}^{-1}V_{21}\right\|_{L_\infty} < 1$, there exists a $r \in (0,1)$ such that $\left\|V_{22}^{-1}V_{21}\right\|_{L_\infty} = 1 - r$. It follows from Lemma 5.6 that there exists a $\delta_1 > 0$ such that $\delta_1 < \varepsilon$ and for any ζ satisfying $0 < \zeta < \delta_1$,

$$\left\|V_{22}(\zeta + \cdot)^{-1}V_{21}(\zeta + \cdot)\right\|_{L_\infty} \leq 1 - \frac{r}{2}.$$

Similarly, it follows from Lemma 5.6 that there exists a $\delta_2 > 0$ such that $\delta_2 < \varepsilon$ and for any ζ satisfying $0 < \zeta < \delta_2$,

$$\|U_1(\zeta + \cdot)U_2(\zeta + \cdot)^{-1}\|_{L_\infty} \leq 1 + \frac{\frac{r}{4}}{1 - \frac{r}{4}} = \frac{1}{1 - \frac{r}{4}}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$, and fix a ζ satisfying $0 < \zeta < \delta$. We define the following function $\phi : [0,1] \times \mathbb{C}_0 \to \mathbb{C}$ by

$$\phi(\alpha, s) = \det\left(\alpha V_{21}(\zeta + s)U_1(\zeta + s) + V_{22}(\zeta + s)U_2(\zeta + s)\right),$$

where $\alpha \in [0,1]$ and $s \in \mathbb{C}_0$.
a. We know that

$$\begin{aligned}
\phi(0,\cdot) &= \det\left(V_{22}(\zeta + \cdot)U_2(\zeta + \cdot)\right) \text{ and} \\
\phi(1,\cdot) &= \det\left(V_{21}(\zeta + \cdot)U_1(\zeta + \cdot) + V_{22}(\zeta + \cdot)U_2(\zeta + \cdot)\right)
\end{aligned}$$

are analytic in $\mathbb{C}_{+,-\zeta/2}$.
b. $\phi(0,\cdot)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$. This follows since $\det V_{22}$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$ (from Lemma 5.14) and $\det U_2$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$ (see (5.23)).
$\phi(1,\cdot)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$, since V_{21} is strictly proper, U_1 is bounded on $\overline{\mathbb{C}_+}$, and $\phi(0,\cdot)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$.
c. $(\alpha, s) \mapsto \phi(\alpha, s) : [0,1] \times \mathbb{C}_0$ is a continuous function, since it is a composition of continuous functions.
d. We have

$$\begin{aligned}
\phi(\alpha, s) &= \det(V_{22}(\zeta + s)) \det\left(U_2(\zeta + s)\right) \cdot \\
&\quad \det\left(I + \alpha V_{22}(\zeta + s)^{-1}V_{21}(\zeta + s)U_1(\zeta + s)U_2(\zeta + s)^{-1}\right) \\
&\neq 0,
\end{aligned}$$

since

$$\begin{aligned}&\left\|\alpha V_{22}(\zeta+\cdot)^{-1}V_{21}(\zeta+\cdot)U_1(\zeta+\cdot)U_2(\zeta+\cdot)^{-1}\right\|_\infty\\ &\quad\le\; 1\;\left\|V_{22}(\zeta+\cdot)^{-1}V_{21}(\zeta+\cdot)\right\|_\infty\left\|U_1(\zeta+\cdot)U_2(\zeta+\cdot)^{-1}\right\|_\infty\\ &\quad\le\;\left[1-\frac{r}{2}\right]\frac{1}{1-\frac{r}{4}}<1,\end{aligned}$$

$\det(V_{22}(\zeta+s))\neq 0$ and $\det(U_2(\zeta+s))\neq 0$.
e. $\phi(\alpha,\infty)\neq 0$, since V_{21} is strictly proper, U_1 is bounded on $\overline{\mathbb{C}_+}$, and $\det V_{22}\det U_2$ has a nonzero limit at infinity in $\overline{\mathbb{C}_+}$.
Thus the function ϕ satisfies the assumptions of Lemma 5.9, and hence it follows that the Nyquist indices of $\phi(0,\cdot)$ and $\phi(1,\cdot)$ are the same. Since $\phi(0,\cdot)$ and $\phi(1,\cdot)$ are analytic in $\mathbb{C}_{+,-\frac{\zeta}{2}}$, this implies that the number of zeros are the same and so the sum of the number of zeros of $s\mapsto\det(V_{22}(\zeta+s))$ in $\overline{\mathbb{C}_+}$ plus the number of zeros of $s\mapsto\det(U_2(\zeta+s))$ in $\overline{\mathbb{C}_+}$ equals the number of zeros of $s\mapsto\det(V_{21}(\zeta+s)U_1(\zeta+s)+V_{22}(\zeta+s)U_2(\zeta+s))$. By (5.21) we see that this equals the number of zeros of $\det(M(\zeta+s))$ in $\overline{\mathbb{C}_+}$.

In particular, we obtain that the number of zeros of $s\mapsto\det(V_{22}(\zeta+s))$ in $\overline{\mathbb{C}_+}$ is less than or equal to l. But since the choice of ζ can be made arbitrarily small, it follows that $\det(V_{22})$ has at most l zeros in $\overline{\mathbb{C}_+}$. Thus $V_{22}^{-1}\in\hat{\mathcal{A}}_l^{m\times m}$, which means that $l_*\le l$.

Step 4. We prove that $\Lambda_{11}^{-1}\in\hat{\mathcal{A}}_l^{k\times k}$.
Since $V=\Lambda^{-1}$, it can be easily checked (by block matrix manipulation) that

$$\Lambda_{11}^{-1}=V_{11}-V_{12}V_{22}^{-1}V_{21}$$

and so it follows from Lemma 5.7 that $\Lambda_{11}^{-1}\in\hat{\mathcal{A}}_l^{k\times k}$. ∎

Lemma 5.16 *Suppose that there exists a $\Lambda\in\hat{\mathcal{A}}^{(k+m)\times(k+m)}$ such that S3, S4, S5 and S6 from Theorem 5.11 hold, and let $V=\Lambda^{-1}$. Then $K_0:=V_{12}V_{22}^{-1}$ is a solution for the sub-optimal Hankel norm approximation problem, and $K_0\in\hat{\mathcal{A}}_l^{k\times m}$.*

Proof: The proof follows similar lines as in Sasane and Curtain [65]. We have that the assumptions S1-S6 hold. From S6 we have that $\Lambda_{11}^{-1}\in\hat{\mathcal{A}}_l^{k\times k}$. Since $\Lambda^{-1}=V$ it can be easily checked that

$$V_{22}^{-1}=\Lambda_{22}-\Lambda_{21}\Lambda_{11}^{-1}\Lambda_{12}.$$

Using Lemma 5.7 it follows that $V_{22}^{-1}\in\hat{\mathcal{A}}_l^{m\times m}$. We know that $V_{12}\in\hat{\mathcal{A}}^{k\times m}$. Applying Lemma 5.7 once more, we obtain that $K_0\in\hat{\mathcal{A}}_l^{m\times m}$.

We have the following equalities:

$$\begin{aligned}\begin{bmatrix}G(s)+K_0(s)\\ I_m\end{bmatrix}&=\begin{bmatrix}I_k & G(s)\\ 0 & I_m\end{bmatrix}\begin{bmatrix}K_0(s)\\ I_m\end{bmatrix}\\ &=\begin{bmatrix}I_k & G(s)\\ 0 & I_m\end{bmatrix}V(s)\begin{bmatrix}0\\ V_{22}(s)^{-1}\end{bmatrix}\end{aligned}\tag{5.24}$$

for all $s \in \mathbb{C}_0$. So, on the imaginary axis holds

$$
\begin{aligned}
&(G+K_0)^*(G+K_0) - \sigma^2 I_m \\
&\quad = \begin{bmatrix} G+K_0 \\ I_m \end{bmatrix}^* J_{\sigma,k,m} \begin{bmatrix} G+K_0 \\ I_m \end{bmatrix} \\
&\quad \overset{(5.24)}{=} \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^* V^* \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix}^* J_{\sigma,k,m} \begin{bmatrix} I_k & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^* \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}.
\end{aligned}
$$

We have used (5.6) from S3 and the fact that V is the inverse of Λ. Considering the first and the last terms of the previous sequence of equalities, we have that

$$\|(G+K_0)(s)u\|^2 - \sigma^2\|u\|^2 = -\|V_{22}(s)^{-1}u\|^2 \tag{5.25}$$

for all $u \in \mathbb{C}^m$ and $s \in \mathbb{C}_0$. From (5.25) we have that

$$\|(G+K_0)(s)u\|^2 - \sigma^2\|u\|^2 \leq 0$$

for all $u \in \mathbb{C}^m$. Using this inequality and (5.25) we obtain that

$$\|G+K_0\|_\infty \leq \sigma.$$

■

Remark 5.17 *Since $K_0 = V_{12}V_{22}^{-1}$, with $V_{12} \in \hat{\mathcal{A}}$, and $V_{22}^{-1} \in \hat{\mathcal{A}}_{[r]}^{m\times m}$ for some $r \in \mathbb{N}$, we have that $K_0 \in \hat{\mathcal{A}}_{[r]}^{k\times m}$.*

We are able to prove now the main theorem.

Proof: of Theorem 5.12.

1 ⇒ 2 : Consider $G^\sim \in \hat{\mathcal{A}}^{m\times k}$ and let σ be such that $\sigma \neq \sigma_p$ for $p \in \mathbb{N}$. Suppose that there exists a $K \in \hat{\mathcal{A}}_l^{k\times m}$ such that $\|G+K\|_{L_\infty} \leq \sigma$. The proof is a consequence of Lemma 5.13 and Lemma 5.15.

2 ⇒ 1 : This implication follows form Lemma 5.16.

It remains to be proved that all solutions to the sub-optimal Hankel norm approximation problem are given by

$$K(s) = X_1(s)X_2(s)^{-1},$$

where X_1 and X_2 are given by (5.11), namely

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix}.$$

for some $Q \in \hat{\mathcal{A}}^{k\times m}$ satisfying $\|Q\|_{H_\infty} \leq 1$.

a. Provided that either 1. or 2. holds, we prove that any $K = X_1 X_2^{-1}$ as above is a solution for the sub-optimal Hankel norm approximation problem.

Take $Q \in \hat{\mathcal{A}}^{k\times m}$ satisfying $\|Q\|_{H_\infty} \leq 1$, X_1 and X_2 given by (5.11). Using Lemma 5.14.2 we obtain that $\|V_{22}(s)^{-1}V_{21}(s)Q(s)\| < 1$ for all $s \in \mathbb{C}_0$. Since $X_2 = V_{21}Q + V_{22}$ and $V_{22}(s)$ is invertible for all $s \in \mathbb{C}_0$ we have that $X_2(s)$ is also invertible for all $s \in \mathbb{C}_0$.

Using similar equalities as in the proof of Lemma 5.16 we obtain that

$$\begin{aligned}(G+K)^*(G+K) - \sigma^2 I_m &= \\ &= X_2^{-*}\begin{bmatrix} Q \\ I_m \end{bmatrix}^*\begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} Q \\ I_m \end{bmatrix}X_2^{-1} \\ &= X_2^{-*}(Q^*Q - I_m)X_2^{-1}.\end{aligned}$$

Since $\|Q\|_{H_\infty} \leq 1$ we see that $\|G+K\|_\infty \leq \sigma$, so $K = X_1X_2^{-1}$ where X_1 and X_2 are given by (5.11) are solutions for the sub-optimal Hankel norm approximation problem.

We prove now that $K \in \hat{\mathcal{A}}_l$. Since Q is proper, V_{21} is strictly proper, V_{22} is proper and $\det(V_{22})$ has a nonzero limit at infinity it follows that $\det(X_2)$ has a nonzero limit at infinity. Applying the Nyquist Lemma 5.9 as in Step 3. in the proof of Lemma 5.15 for $U_2 = I$ and $U_1 = Q$, we obtain that $\det(X_2)$ and $\det(V_{22})$ have the same number of zeros in the open right half-plane. It follows that $\det(X_2)$ has at most l zeros in $\overline{\mathbb{C}_+}$ and all of them are contained in the open right half-plane. From here we can see that $\det(X_2) \in \hat{\mathcal{A}}_\infty$, and $K = X_1X_2^{-1} \in \hat{\mathcal{A}}_{[r]} \subset \hat{\mathcal{A}}_l$, where r is the number of zeros of $\det(V_{22})$ in the open right half-plane.

b. Suppose that $K \in \hat{\mathcal{A}}_{[r]} \subset \hat{\mathcal{A}}_l$ is a solution for the sub-optimal Hankel norm approximation problem, and suppose that $K = NM^{-1}$ is a coprime factorization of K over $\hat{\mathcal{A}}$, where M is rational and $\det(M) \in \hat{\mathcal{A}}_\infty$ has exactly r zeros in the right half-plane, none of them on the imaginary axis. We will prove that K can be written as $K = X_1X_2^{-1}$ where X_1 and X_2 are given by (5.11).

Consider U_1 and U_2 given by (5.15) from the proof of Lemma 5.15, step 2. In step 3. of the proof of Lemma 5.15, we have proved that $U_2^{-1} \in \hat{\mathcal{A}}_{[q]}$ for some $q \in \mathbb{N}$, and that the number of zeros of $\det(U_2)$ plus the number of zeros of $\det(V_{22})$ in $\overline{\mathbb{C}_+}$ equals the number of zeros of $\det(M)$ in $\overline{\mathbb{C}_+}$. Using Remark 5.17, this is

$$r + q = r,$$

which implies that $q = 0$, so $U_2^{-1} \in \hat{\mathcal{A}}$. From the relation (5.15) it follows that

$$\begin{bmatrix} NU_2^{-1} \\ MU_2^{-1} \end{bmatrix} = \Lambda^{-1}\begin{bmatrix} U_1U_2^{-1} \\ I_m \end{bmatrix}. \tag{5.27}$$

Defining $Q := U_1U_2^{-1}$ we see that $Q \in \hat{\mathcal{A}}$ and $\|Q\|_\infty \leq 1$ (see Lemma 5.15, step 2.). It is clear now that we can take

$$\begin{aligned} X_1 &= NU_2^{-1} \\ X_2 &= MU_2^{-1} \end{aligned}$$

and the proof is completed. ∎

Using (5.5) and Remark 5.17 the following corrolary of Theorem 5.12 can be obtained.

Corollary 5.18 *Consider $G^{\sim} \in \hat{\mathcal{A}}_0^{m\times k}$, and let σ be such that $\sigma \neq \sigma_p$ for all $p \in \mathbb{N}$. Then the assumptions S3-S5 hold, and the following statements are equivalent:*

1. $\sigma_{l+1} < \sigma < \sigma_l$.
2. *There exists a $K \in \hat{\mathcal{A}}_{[l]}^{k\times m}$ such that $\|G+K\|_{L_\infty} \leq \sigma$.*
3. *The matrix-valued function $\Lambda \in \hat{\mathcal{A}}^{(k+m)\times(k+m)}$ satisfies $\Lambda_{11}^{-1} \in \hat{\mathcal{A}}_{[l]}^{k\times m}$.*

5.3 The sub-optimal Hankel norm approximation problem for Pritchard-Salamon systems

In this section we consider the sub-optimal Hankel norm approximation problem for exponentially stable Pritchard-Salamon systems with finite-dimensional input and output spaces. A solution via J-spectral factorization was first presented by Curtain and Ran [19] using the work of Ball and Helton [5], [6]. Then, Sasane and Curtain [66] presented a self-contained solution. Both approaches construct a solution from a given realization of G.

Similarly as for the Nehari problem (see Chapter 4, Section 4.4), the results presented in the previous sections provide a self-contained frequency-domain solution for the sub-optimal Hankel norm approximation problem for exponentially stable Pritchard-Salamon systems with finite-dimensional input and output spaces. Using Lemma 2.48, this result can be seen as consequences of the Theorem 5.12. However, specific for the state-space approach is the representation (4.32) (the same form as for the Nehari problem) of the J-spectral factor used to described all solutions of the sub-optimal Hankel norm approximation problem.

5.4 The optimal Hankel norm approximation problem

Let $G \in \hat{\mathcal{W}}^{k\times m}$. The problem of finding a $K \in \hat{\mathcal{A}}_l$ that achieves the minimum distance from G to $\hat{\mathcal{A}}_l$, i.e.,

$$\inf_{K\in\hat{\mathcal{A}}_l^{k\times m}} \|G+K\|_{L_\infty} = \sigma_{l+1}.$$

is called the optimal Hankel norm approximation problem (for the Wiener class). The following result would be the natural extension of Theorem 4.16.

Theorem 5.19 *Suppose that $G \in \hat{\mathcal{W}}^{k\times m}$. Then any $K_0 \in H_{\infty,l}^{k\times m}$ solving the optimal Hankel norm approximation problem, that is, satisfying*

$$\|G + K_0\|_\infty = \sigma_{l+1}, \tag{5.28}$$

also satisfies

$$(G + K_0)u_2 = H_G u_2, \tag{5.29}$$

where u_2 is an eigenvector for the compact nonnegative operator $H_G^* H_G$ corresponding to the eigenvalue σ_{l+1}.

A proof for the scalar case was given by Adamjan, Arov and Krein [1]. The solution for the discrete-time, rational matrix-valued functions was first given by the engineers Kung and Lin [46]. We refer also to the paper of Young [79], where this result is stated and nice examples are presented.

A similar statement as in Theorem 4.17 is also true.

Theorem 5.20 *Suppose that $G \in \hat{\mathcal{W}}^{k\times m}$ is given, and that $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in H_2$ is an equalizing vector for the matrix-valued function $W(s)$, defined in (4.12), corresponding to the singular value σ_{l+1}. If there exists a solution K_0 of the optimal Hankel norm approximation problem, then on the imaginary axis it satisfies*

$$K_0 u_2 = u_1. \tag{5.30}$$

Chapter 6

H_∞ sub-optimal control problem

Introduction

The standard H_∞ control problem was introduced in 1984 by J.C. Doyle [24]. In a few words the H_∞ sub-optimal control problem is to find a controller which stabilizes a given plant and which makes the H_∞ norm of the associated closed loop transfer function less than a given positive real number.

Nowadays, there are different techniques for solving the standard H_∞ control problem. We will use coprime factorizations for solving this problem. The idea of factorizing the transfer function of a (not necessarily stable) system as a ratio of two stable transfer functions was first introduced in 1972 by Vidyasagar [73]. Using this (coprime) factorization Green et al. [32, 33] showed that for rational transfer functions the standard H_∞ control problem can be solved if and only if two J-spectral factorizations are solvable. This result has been extended to the infinite-dimensional case for systems in state-space form $\Sigma(A, B, C, D)$, with B and C bounded, by Curtain and Rodriguez [20].

A generalization of the results presented by Green [32] was found by Curtain and Green [15] for a class of systems with irrational transfer matrices, using a result of Ball and Helton [5, 6]. For infinite-dimensional (weakly regular) well-posed linear systems, the standard H_∞ sub-optimal control problem was solved by Mikkola in [52].

A similar result as in Green et al. [32, 33] was proved, also for the rational case, by Meinsma [49], using different techniques. In this chapter we show that this approach extends (partially) to a large class of infinite-dimensional systems, namely to systems having their transfer functions in the quotient field of H_∞ (Nevanlinna class of transfer functions). We prove that if two J-spectral factorizations have a solution, then the standard H_∞ sub-optimal control problem has a solution. Furthermore, we parametrize the set of all solutions. The elements of the Nevanlinna class need not be well-posed in the formulation of Mikkola [52]. However, here we consider only the case of finite-dimensional input and output spaces.

For systems with a state-space realization, the existence of a J-spectral factorization is often equivalent to the existence of a stabilizing solution to an algebraic Riccati equation. This is shown by Green [32] for the finite-dimensional situation, by van Keulen [71] for a class of infinite-dimensional systems, and by Mikkola, for well-posed systems, in [52]. The solution of the algebraic Riccati equation is used to present the state-space formula for all solutions to the H_∞ control problem.

This chapter basically generalizes the proofs of Meinsma as presented in [49]. We do not make use of a state-space representation of our system, neither do we use Riccati equations. Since there are many systems for which the given state-space realization is not well-posed, and thus the state-space approach using Riccati equations is not guaranteed to work, our approach can be applied to a larger class of systems. Furthermore, we expect that our approach will have some numerical advantages. We remark that our class of systems does include dead-time systems. The procedure for solving the J-spectral factorization, as presented in Chapter 3, can be applied to systems that can be approximated by rational functions. However, this procedure does not apply to a pure delay.

In the sixth section of this chapter, we provide a self-contained solution to the standard H_∞ sub-optimal control problem for systems with the transfer function in $\hat{\mathcal{B}}_-$, the class of infinite-dimensional systems considered in Curtain and Green [15]. So, Section 6.6 provides an independent proof of the results in [15] which used the abstract theory of Ball and Helton [5] to obtain the Nehari result. However, if one looks for the result quoted from [5] in [15], one realizes that this is not an obvious corollary of the very abstract and general theory in [5]. This approach we consider in Section 6.6 can also be taken for $\hat{\mathcal{B}}_0$, a subset of the quotient field of the Wiener algebra on the right half-plane.

6.1 Preliminaries

6.1.1 Inner and J-lossless matrices

The concept of inner and J-lossless matrices are closely related as will be shown in Lemma 6.6. We begin with the definition of inner matrices.

Definition 6.1 *A matrix-valued function $G \in H_\infty^{n\times m}$ is said to be* inner *if*

$$G^\sim(s)G(s) = I_m \tag{6.1}$$

for almost all $s \in \mathbb{C}_0$.

Observe that if G is inner, then $n \geq m$. In other words, the matrix is "tall".

Lemma 6.2 *A matrix-valued function $G \in H_\infty^{n\times m}$ is inner if and only if*

$$G^*(s)G(s) \leq I_m \tag{6.2}$$

for almost all $s \in \overline{\mathbb{C}_+}$, with equality on the imaginary axis.

Proof: *Necessity:* If (6.2) is an equality almost everywhere on the imaginary axis, then obviously the condition from the definition of inner matrices is satisfied. So G is inner.

Sufficiency: For the other implication suppose that the matrix-valued function $G \in H_\infty^{n\times m}$ is inner, which means that the equality (6.1) is satisfied almost everywhere on the imaginary axis, so

$$\operatorname*{ess\,sup}_{s\in\mathbb{C}_0} \|G(s)\| = 1.$$

Since $G \in H_\infty^{n\times m}$, we have that

$$\sup_{s\in\mathbb{C}_+} \|G(s)\| = \operatorname*{ess\,sup}_{s\in\mathbb{C}_0} \|G(s)\|$$

which implies that

$$\|G(s)\|^2 \leq 1 \text{ for almost all } s \in \overline{\mathbb{C}_+}.$$

This means that all the singular values of the matrix $G(s)^*G(s)$ are less or equal to one for every fixed value of s. It follows that

$$G(s)^*G(s) - I_m \leq 0, \text{ for almost all } s \in \overline{\mathbb{C}_+},$$

with equality on the imaginary axis. ■

Recall that we made the following notation:

$$J_{\gamma,m,n} := \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_n \end{bmatrix}$$

for $m, n \in \mathbb{N}$ and some positive γ. We write $J_{m,n}$ whenever $\gamma = 1$. We introduce the notions of J-lossless and co-J-lossless matrix-valued functions.

Definition 6.3 *A matrix $G \in H_\infty^{(n+p)\times(m+p)}$ is $J_{\gamma,n,p}$-*lossless *(or $J_{\gamma,n,p}$-inner) if*

$$G(s)^* J_{\gamma,n,p} G(s) \leq J_{m,p} \textit{ for almost all } s \in \overline{\mathbb{C}_+}$$

with equality almost everywhere on the imaginary axis. This means that

$$G^\sim(s) J_{\gamma,n,p} G(s) = J_{m,p}, \quad \textit{for almost all } s \in \mathbb{C}_0, \tag{6.3}$$

$$G(s)^* J_{\gamma,n,p} G(s) \leq J_{m,p}, \quad \textit{for all } s \in \mathbb{C}_+. \tag{6.4}$$

Notice that a J-lossless matrix need not to be square; it is "tall" ($n \geq m$), and it is assumed to be partitioned so that the lower-right corner is square.

Definition 6.4 *A partitioned matrix $G \in H_\infty^{(m+p)\times(n+p)}$ is* co-$J_{\gamma,m,p}$-lossless *if it satisfies*

$$G(s) J_{\gamma,m,p} G^\sim(s) = J_{n,p}, \quad \textit{for almost all } s \in \mathbb{C}_0, \tag{6.5}$$

$$G(s) J_{\gamma,m,p} G(s)^* \leq J_{n,p}, \quad \textit{for all } s \in \mathbb{C}_+. \tag{6.6}$$

Remark 6.5 *It is easy to see that a matrix-valued function $G \in L_\infty$ is J-lossless if and only if $G(s)^T$ is co-J-lossless.*

The following result describes the relation between inner and J-lossless matrix-valued functions.

Lemma 6.6 *Suppose that n_u, n_y and n_z are given positive integers. In what follows w and y are elements in $H_2^{n_y}$, $z \in H_2^{n_z}$ and $u \in H_2^{n_u}$. We have the following relation between inner and J-lossless.*

1. *If $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in H_\infty^{(n_z+n_y)\times(n_u+n_y)}$ (with M partitioned compatibly) is J_{n_z,n_y}-lossless, then M_{22} is invertible over H_∞ and*
$$\left\{(z,w,u,y) \mid \begin{bmatrix} z \\ w \end{bmatrix} = M \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} u \\ y \end{bmatrix} \in H_2^{n_u+n_y}\right\} \tag{6.7}$$
equals
$$\left\{(z,w,u,y) \mid \begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \begin{bmatrix} w \\ u \end{bmatrix} \in H_2^{n_y+n_u}\right\}, \tag{6.8}$$
where G is the matrix defined as
$$G = \begin{bmatrix} M_{12}M_{22}^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \\ M_{22}^{-1} & -M_{22}^{-1}M_{21} \end{bmatrix}. \tag{6.9}$$
Moreover, G is inner.

2. *If $G \in H_\infty^{(n_z+n_y)\times(n_y+n_u)}$ is an inner matrix whose lower left $n_y \times n_y$ block element is in $GH_\infty^{n_y\times n_y}$, then there exists a unique stable J_{n_u,n_y}-lossless M for which the sets given in (6.7) and (6.8) coincide. Moreover, if G is given as*
$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$
then M is given as
$$M = \begin{bmatrix} G_{12} - G_{11}G_{21}^{-1}G_{22} & G_{11}G_{21}^{-1} \\ -G_{21}^{-1}G_{22} & G_{21}^{-1} \end{bmatrix}. \tag{6.10}$$

Proof: We begin by proving the first assertion. It is easy to see that for $s \in \mathbb{C}_+$

$$M(s)^* J_{n_u,n_y} M(s) =$$
$$\begin{bmatrix} M_{11}(s)^*M_{11}(s) - M_{21}(s)^*M_{21}(s) & M_{11}(s)^*M_{12}(s) - M_{21}(s)^*M_{22}(s) \\ M_{12}(s)^*M_{11}(s) - M_{22}(s)^*M_{21}(s) & M_{12}(s)^*M_{12}(s) - M_{22}(s)^*M_{22}(s) \end{bmatrix}.$$

Since M is J_{n_u,n_y}-lossless, we have that this, in particular, implies that

$$-M_{22}(s)^*M_{22}(s) \le M_{12}(s)^*M_{12}(s) - M_{22}(s)^*M_{22}(s) \le -I_{n_y} < 0 \tag{6.11}$$

for all $s \in \mathbb{C}_+$. With Lemma 2.7 this implies that M_{22} is invertible over H_∞, and thus G in (6.9) is well-defined.
If (z, w, u, y) is such that $\begin{bmatrix} z \\ w \end{bmatrix} = M \begin{bmatrix} u \\ y \end{bmatrix}$, with $\begin{bmatrix} u \\ y \end{bmatrix} \in H_2$ then, by the stability of M, we have that $\begin{bmatrix} z \\ w \end{bmatrix} \in H_2$. Furthermore,

$$\begin{aligned} z &= M_{11}u + M_{12}y; \\ w &= M_{21}u + M_{22}y, \end{aligned}$$

with M_{22} invertible, implies that

$$y = M_{22}^{-1}w - M_{22}^{-1}M_{21}u \tag{6.12}$$

and

$$z = \left(M_{11} - M_{12}M_{22}^{-1}M_{21}\right)u + M_{12}M_{22}^{-1}w.$$

Thus (z, w, u, y) lies in the set defined by (6.8). The other inclusion can be proved similarly.
It remains to show that G is inner. We show that

$$G^{\sim}(s)G(s) = I \Longleftrightarrow M^{\sim}(s)J_{n_z,n_w}M(s) = J_{n_u,n_y}, \tag{6.13}$$

for almost all $s \in \mathbb{C}_0$. Let $\begin{bmatrix} w \\ u \end{bmatrix} \in \mathbb{C}^{n_y+n_u}$ and define $\begin{bmatrix} z \\ y \end{bmatrix} \in \mathbb{C}^{n_z+n_y}$ as

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s)\begin{bmatrix} w \\ u \end{bmatrix}$$

for almost all $s \in \mathbb{C}_0$. Then $G^{\sim}(s)G(s) = I$ implies that

$$\|z\|^2 + \|y\|^2 = \|w\|^2 + \|u\|^2 .$$

This is equivalent to

$$\|z\|^2 - \|w\|^2 = \|u\|^2 - \|y\|^2 .$$

Since M is J_{n_z,n_y}-lossless, we see that this is equivalent to

$$\begin{bmatrix} u^* & y^* \end{bmatrix} M(s)^* J_{n_z,n_y} M(s) \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} u^* & y^* \end{bmatrix} J_{n_u,n_y} \begin{bmatrix} u \\ y \end{bmatrix}$$

for almost all $s \in \mathbb{C}_0$. This holds for all $\begin{bmatrix} u \\ y \end{bmatrix}$ for which there exist a w such that $\begin{bmatrix} z \\ y \end{bmatrix} = G(s)\begin{bmatrix} w \\ u \end{bmatrix}$. From (6.12), we see that this set equals $\mathbb{C}^{n_u+n_y}$. In other words (6.13) is shown. In the same way it can be proved the equivalence

$$G^*(s)G(s) \leq I \Longleftrightarrow M^*(s)J_{n_z,n_w}M(s) \leq J_{n_u,n_y}, \text{ for } s \in \mathbb{C}_+. \tag{6.14}$$

Using now Lemma 6.2 we have proved the first assertion. For the second assertion we define the matrix M as

$$M = \begin{bmatrix} G_{12} - G_{11}G_{21}^{-1}G_{22} & G_{11}G_{21}^{-1} \\ -G_{21}^{-1}G_{22} & G_{21}^{-1} \end{bmatrix}. \tag{6.15}$$

The proof follows now the opposite direction. ∎

In order to check whether a matrix-valued function is J-lossless, one can replace condition (6.4) by an equivalent condition.

Lemma 6.7 *A partitioned matrix-valued function $M \in H_\infty^{(n+p)\times(m+p)}$ is $J_{\gamma,n,p}$-lossless if and only if the following two conditions are satisfied*

1. $M^\sim(s)J_{\gamma,n,p}M(s) = J_{m,p}$, *for almost all* $s \in \mathbb{C}_0$;
2. *the lower right* $p \times p$ *block element of the matrix* M *is invertible over* $H_\infty^{p\times p}$.

Proof: Let M be partitioned as follows

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and define

$$M_\gamma := \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix} M = \begin{bmatrix} M_{11} & M_{12} \\ \gamma M_{21} & \gamma M_{22} \end{bmatrix}.$$

From Lemma 6.6 it follows that M_γ is $J_{m,p}$-lossless if and only if the matrix G_γ defined as

$$G_\gamma = \begin{bmatrix} M_{12}M_{22}^{-1}\gamma^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \\ M_{22}^{-1}\gamma^{-1} & -M_{22}^{-1}M_{21} \end{bmatrix}.$$

is inner and its lower-left block, i.e. M_{22}^{-1} is in H_∞. Using now the definition of an inner matrix and the equivalence (6.13) we have that the matrix-valued function G_γ is inner is equivalent with

$$M_\gamma^\sim(s)J_{n,p}M_\gamma(s) = J_{m,p},$$

for almost all $s \in \mathbb{C}_0$, which is equivalent with

$$M^\sim(s)J_{\gamma,n,p}M(s) = J_{m,p},$$

for almost all $s \in \mathbb{C}_0$. So, the equivalence is proved. ∎

For co-J-lossless we have a similar result. The proof is easy by using Remark 6.5.

Lemma 6.8 *A partitioned matrix-valued function $M \in H_\infty$ is co-$J_{\gamma,q,p}$-lossless if and only if the following two conditions are satisfied*

1. $M(s)J_{\gamma,m,p}M^\sim(s) = J_{n,p}$, *for almost all* $s \in \mathbb{C}_0$;
2. *the lower right* $p \times p$ *block element of* M *is invertible over* $H_\infty^{p\times p}$.

6.1.2 Positivity results

In this last subsection we will present some results that will be useful in the sequel. The important issue is the connection between stability and positivity.

Definition 6.9 *Consider the Hardy space $H_2^{m+n,\perp}$ with the inner product $\langle\cdot,\cdot\rangle$. A subspace B of $H_2^{m+n,\perp}$ is* positive *with respect to the $J_{\gamma,m,n}$-*inner product

$$[f,g] := \langle f, J_{\gamma,m,n} g\rangle, \ f,g \in H_2^{m+n,\perp},$$

if for every $x \in B$

$$\langle x, J_{\gamma,m,n} x\rangle \geq 0.$$

It is strictly positive *with respect to the $J_{\gamma,m,n}$-inner product if there exists an $\varepsilon > 0$ such that for all non-zero $x \in B$ holds*

$$\langle x, J_{\gamma,m,n} x\rangle \geq \varepsilon\langle x, x\rangle.$$

Definition 6.10 *Given a stable matrix-valued function $G \in H_\infty^{n\times m}$, define*

$$B_G = \{x \in H_2^{\perp,m} \mid Gx \in H_2^n\}. \tag{6.18}$$

We say that B_G is the set generated by *G.*

Note that the set generated by G is a subset of $H_2^\perp$. The next lemma shows that the set generated by G is invariant under premultiplication of G by a bistable matrix-valued function.

Lemma 6.11 *Consider two stable matrix-valued functions G, $\overline{G} \in H_\infty^{n\times m}$. If there exists a $W \in GH_\infty^{n\times n}$ such that $\overline{G} = WG$, then G and $\overline{G}$ generate the same set, i.e., $B_G = B_{\overline{G}}$.*

Proof: Suppose that there exists a $W \in GH_\infty^{n\times n}$ such that $\overline{G} = WG$. We have to prove that $B_G = B_{\overline{G}}$. Since $W \in GH_\infty^{n\times n}$ we know that $WH_2^n = H_2^n$ (see Francis [27]). From the definition of B_G, we have that $GB_G \subset H_2^n$ and so

$$WGB_G \subset WH_2^n = H_2^n,$$

which implies that

$$W(GB_G) \subset H_2^n.$$

However,

$$B_{WG} = \{x \in H_2^{\perp,m} \mid WGx \in H_2^n\},$$

which implies that

$$B_G \subset B_{WG}.$$

Using that $\overline{G} = WG$ we get $B_G \subset B_{\overline{G}}$. The other inclusion can be proved similarly by using the equality $G = W^{-1}\overline{G}$. ■

Lemma 6.12 *Let F_1 and F_2 be two stable matrix-valued functions such that F_2 is invertible over $H_\infty^{p\times p}$ and $\|F_2^{-1}F_1\|_{H_\infty} \leq \gamma$, where γ is a positive real number. Then the set generated by $[F_1\ F_2]$, $B_{[F_1 F_2]} \subset H_2^{m+p,\perp}$, is positive with respect to the $J_{\frac{1}{\gamma},m,p}$ inner product.*
If we have strictly inequality, i.e., $\|F_2^{-1}F_1\|_{H_\infty} < \gamma$, then $B_{[F_1 F_2]} \subset H_2^{m+p,\perp}$ is strictly positive with respect to the $J_{\frac{1}{\gamma},m,p}$-inner product.

Proof: Since F_2 is invertible over H_∞, we have that

$$\begin{bmatrix} F & I_p \end{bmatrix} = F_2^{-1} \begin{bmatrix} F_1 & F_2 \end{bmatrix},$$

where $F := F_2^{-1}F_1$. Applying Lemma 6.11, the following equality holds

$$B_{\begin{bmatrix} F_1 & F_2 \end{bmatrix}} = B_{\begin{bmatrix} F & I_p \end{bmatrix}}.$$

Let $w \in B_{\begin{bmatrix} F & I_p \end{bmatrix}}$ be partitioned as $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, compatibly with the partition of $\begin{bmatrix} F & I_p \end{bmatrix}$. From the definition of $B_{\begin{bmatrix} F & I_p \end{bmatrix}}$ it follows that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in H_2^{m+p,\perp} \text{ and } \begin{bmatrix} F & I_p \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in H_2^p$$

which is equivalent to

$$w_1 \in H_2^{m,\perp} \text{ and } w_2 = -Q(Fw_1),$$

where Q is the projection from $L_2^p(\mathbb{C}_0)$ onto $H_2^{p,\perp}$.
We have

$$\begin{aligned}
\langle w, J_{\frac{1}{\gamma},q,p} w\rangle &= \langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, J_{\frac{1}{\gamma},m,p} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \rangle = \|w_1\|_{L_2}^2 - \frac{1}{\gamma^2}\|w_2\|_{L_2}^2 \\
&= \|w_1\|_{L_2}^2 - \frac{1}{\gamma^2}\|Q(Fw_1)\|_{L_2}^2 \\
&\geq \|w_1\|_{L_2}^2 (1 - \frac{1}{\gamma^2}\|F\|_{H_\infty}^2) \\
&= \frac{1}{\gamma^2}\|w_1\|_{L_2}^2 (\gamma^2 - \|F_2^{-1}F_1\|_{H_\infty}) \geq 0.
\end{aligned} \tag{6.19}$$

Hence, we proved that for every $w \in B_{\begin{bmatrix} F & I_p \end{bmatrix}}$ there holds $\langle w, J_{\frac{1}{\gamma},m,p} w\rangle \geq 0$, which means that $B_{\begin{bmatrix} F & I_p \end{bmatrix}}$ is positive in the J-inner product.
The implication for the strictly positive case follows directly from (6.19) and using that $\|F_2^{-1}F_1\| < \gamma$. ■

6.2 Problem formulation

The standard H_∞-control problem diagram is shown in Figure 6.1, where w, u, z and y are vector valued signals:

- w is the exogenous input,
- u is the control signal,
- z is the output to be controlled,
- y is the measured output.

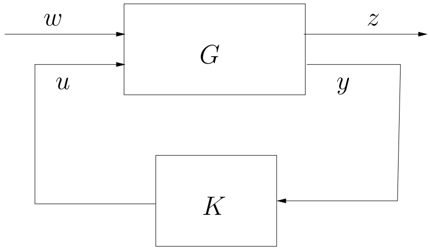

Figure 6.1: The standard H_∞-control problem diagram.

The transfer matrices G and K are assumed to be in F_∞. Furthermore, we assume that G is a stabilizable plant which is decomposed as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{6.20}$$

In the closed loop we consider z, y and u as outputs. The diagram of Figure 6.1 leeds to the following algebraic equations

$$\begin{cases} z = G_{11}w + G_{12}u \\ y = G_{21}w + G_{22}u \\ u = Ky \end{cases}. \tag{6.21}$$

In order to define stability we consider the diagram given in Figure 6.2, where two additional signals, v_1 and v_2, are introduced. The algebraic equations corresponding to Figure 6.2 can be represented in the following block format

$$\begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -K & I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{6.22}$$

For well posedness we require that the matrix-valued function on the left side is invertible. This is

$$T^{-1} = \begin{bmatrix} I & G_{12}K(I-G_{22}K)^{-1} & G_{12} + G_{12}K(I-G_{22}K)^{-1}G_{22} \\ 0 & (I-G_{22}K)^{-1} & (I-G_{22}K)^{-1}G_{22} \\ 0 & K(I-G_{22}K)^{-1} & I + K(I-G_{22}K)^{-1}G_{22} \end{bmatrix} \tag{6.23}$$

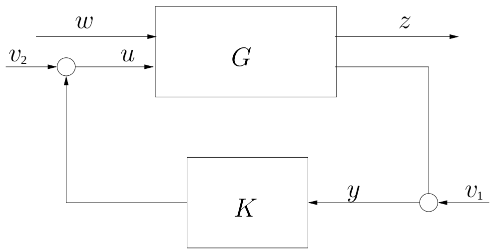

Figure 6.2: Stability diagram.

exists for almost all $s \in \mathbb{C}_+$. A sufficient condition for this to hold is that $\det(I - G_{22}(s)K(s)) \neq 0$ for almost all s in the right half-plane.

For the feedback system of Figure 6.2, we define input-output stability.

Definition 6.13 *The feedback system (G, K) of Figure 6.2, where $G, K \in F_\infty$, is said to be input-output stable if the following holds:*

1. $\det(I - G_{22}(s)K(s)) \neq 0$ *for all s in the open right half plane;*
2. *The transfer functions $S = (I - G_{22}K)^{-1}$, KSG_{22}, $G_{11} + G_{12}KSG_{21}$, KSG_{21}, $G_{12}KS$, KS, SG_{22}, SG_{21} and $G_{12}(I + KSG_{22})$ are in H_∞.*

Remark 6.14 *If the feedback system (G, K) of Figure 6.2 is input-output stable, then*

1. $T^{-1} \in H_\infty$,
2. $T^{-1} \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} \in H_\infty$.

This follows from Definition 6.13, item 2, the expression (6.23) of T^{-1}, and the fact that

$$T^{-1} \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} G_{11} + G_{12}KSG_{21} & G_{12}KS & G_{12} + G_{12}KSG_{22} \\ SG_{21} & S & SG_{22} \\ KSG_{21} & KS & I + KSG_{22} \end{bmatrix}.$$

Next we define the optimal and the sub-optimal H_∞ control problem.

Definition 6.15 (The standard H_∞ optimal control problem) *Given a stabilizable plant $G \in F_\infty$ minimize the H_∞ norm of the transfer function T_{zw}, from w to z, over all stabilizable controllers $K \in F_\infty$. Moreover, find*

$$\gamma_{opt} = \inf_K \|T_{zw}\|_{H_\infty}. \tag{6.24}$$

Definition 6.16 (The standard H_∞ sub-optimal control problem) *Given a stabilizable plant $G \in F_\infty$ and the positive bound γ, find a compensator $K \in F_\infty$ such that the transfer function T_{zw}, from w to z, satisfies*

$$\|T_{zw}\|_{H_\infty} < \gamma.$$

If it is possible, describe the general form of all stabilizable controllers which satisfies the above inequality.

Notice that the transfer matrix from w to z is

$$T_{zw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

We will reformulate the standard H_∞-control problem using coprime factorizations. From Theorem 2.35 we know that every stabilizable G possesses a left-coprime factorization. Let G be a stabilizable plant and let

$$G = D^{-1}N \tag{6.25}$$

be a left-coprime factorization of G over H_∞, where

$$\begin{aligned} D &\in H_\infty^{(n_z+n_y)\times(n_z+n_y)}, \\ N &\in H_\infty^{(n_z+n_y)\times(n_w+n_u)}. \end{aligned}$$

$D = [D_1\ D_2]$ and $N = [N_1\ N_2]$ corresponds to the partitioning of the output and the input of G, where

$$\begin{aligned} D_1 &\in H_\infty^{(n_z+n_y)\times n_z}, D_2 \in H_\infty^{(n_z+n_y)\times n_y}, \\ N_1 &\in H_\infty^{(n_z+n_y)\times n_w}, N_2 \in H_\infty^{(n_z+n_y)\times n_u}. \end{aligned}$$

Since K stabilizes G if and only if G stabilizes K (see Smith [68]), we have from Theorem 2.35 that K possesses a coprime factorization. Let $K = K_d^{-1}K_n$ be a left-coprime factorization over H_∞ of the controller, so

$$K_d \in H_\infty^{n_u\times n_u},\ K_n \in H_\infty^{n_u\times n_y}.$$

From Figure 6.1 we see that we can describe the closed-loop signal equation of the system as

$$\begin{bmatrix} -N_1 & D_1 & -N_2 & D_2 \\ 0 & 0 & K_d & -K_n \end{bmatrix} \begin{bmatrix} w \\ z \\ u \\ y \end{bmatrix} = 0. \tag{6.28}$$

The upper row block

$$-N_1w + D_1z - N_2u + D_2y = 0 \tag{6.29}$$

defines the plant, and the lower row block

$$K_du - K_ny = 0 \tag{6.30}$$

defines the controller.
In this case the extended closed-loop of Figure 6.2 has the equations:

$$\begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 & D_2 & 0 \\ 0 & 0 & K_d \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{6.31}$$

Let us denote

$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}, \; B = \begin{bmatrix} N_1 & D_2 & 0 \\ 0 & 0 & K_d \end{bmatrix}. \tag{6.32}$$

Lemma 6.17 *Suppose that $G \in F_\infty$ is given in the form (6.20) and that G has the left-coprime factorization over H_∞ as described before (see (6.25)). Then the following statements are equivalent:*

1. *The feedback system (G, K) of Figure 6.2 is input-output stable for the controller $K \in F_\infty$.*
2. *The controller $K \in F_\infty$ has a left-coprime factorization $K = K_d^{-1} K_n$ over H_∞ and the matrix*
$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \tag{6.33}$$
is invertible over H_∞.

Proof: <u>1⇒2:</u> Suppose that the feedback system (G, K) of Figure 6.2 is input-output stable for the controller $K \in F_\infty$ and that the left-coprime factorization over H_∞ of K is given as $K = K_d^{-1} K_n$. Denote

$$T := \begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -K & I \end{bmatrix}, \quad \text{and } S := \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{6.34}$$

The inverse of T is

$$T^{-1} = \begin{bmatrix} I & G_{12}K(I - G_{22}K)^{-1} & G_{12} - G_{12}K(I - G_{22}K)^{-1} \\ 0 & (I - G_{22}K)^{-1} & (I - G_{22}K)^{-1}G_{22} \\ 0 & K(I - G_{22}K)^{-1} & I + K(I - G_{22}K)^{-1}G_{22} \end{bmatrix}.$$

Since the feedback system (G, K) is input-output stable, it follows that $T^{-1} \in H_\infty$ (see Remark 6.14). The loop equations of Figure 6.2 can be written as

$$T \begin{bmatrix} z \\ y \\ u \end{bmatrix} = S \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \tag{6.35}$$

and multiplying this by the following matrix in H_∞

$$Z = \begin{bmatrix} D & 0 \\ 0 & K_d \end{bmatrix}$$

gives

$$A\begin{bmatrix} z \\ y \\ u \end{bmatrix} = B\begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}, \tag{6.36}$$

where we have used that

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}.$$

For $[A\ B]$ the following equality holds

$$\begin{bmatrix} A & B \end{bmatrix} = LR, \tag{6.37}$$

where

$$R = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & I \end{bmatrix} \tag{6.38}$$

and

$$L = \begin{bmatrix} D_1 & D_2 & N_1 & N_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_n & K_d \end{bmatrix}. \tag{6.39}$$

Since D, N and K_d, K_n are respectively left-coprime there exist the matrices E_1, F_1, E_2 and F_2 such that

$$DE_1 + NF_1 = I$$

and

$$K_nE_2 + K_dF_2 = I.$$

Denote

$$\begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} R^{-1} = L. \tag{6.40}$$

We have

$$\begin{bmatrix} X & Y \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} = I \tag{6.41}$$

and using (6.40) it becomes

$$\begin{bmatrix} A & B \end{bmatrix} R^{-1}\begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} = I. \tag{6.42}$$

So A and B are left-coprime; namely, there exists a Q in H_∞,

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = R^{-1} \begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} \tag{6.43}$$

such that

$$AQ_1 + BQ_2 = I. \tag{6.44}$$

From this we see that the inverse of A (it exists since $A = ZT$ and both of them are invertible) is given by

$$A^{-1} = Q_1 + A^{-1}BQ_2. \tag{6.45}$$

It remains to prove that $A^{-1} \in H_\infty$. We know that

$$A^{-1}B = (ZT)^{-1}(ZS) = T^{-1}S$$

with $T^{-1}S \in H_\infty$ (see Remark 6.14), hence $A^{-1}B \in H_\infty$. Recalling the equality (6.45) and the fact that $Q_1, Q_2 \in H_\infty$, we obtain that $A^{-1} \in H_\infty$. So A is invertible over H_∞.

2⇒1: We have that the controller $K \in F_\infty$ has a left-coprime factorization $K = K_d^{-1}K_n$ over H_∞ and the matrix A is invertible over H_∞. The closed loop equations (6.31) can be written as

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = A^{-1}B \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

with $A^{-1}B \in H_\infty$, so K is a stabilizing controller for G. ■

Now we can reformulate the standard H_∞ sub-optimal control problem into an equivalent problem involving the coprime factorization of G and K.

Definition 6.18 (The standard H_∞ sub-optimal control problem) *Given a transfer matrix* $\begin{bmatrix} -N_1 & D_1 & -N_2 & D_2 \end{bmatrix} \in H_\infty^{(n_z+n_y)\times(n_w+n_z+n_u+n_y)}$ *find a compensator* $K \in F_\infty^{n_u \times n_y}$ *with a left-coprime factorization* $K_d^{-1}K_n$*, such that the transfer matrix* T_{zw} *(see Figure 6.1) from* w *to* z *induced by the frequency domain equation*

$$A \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} w \tag{6.46}$$

satisfies $\|T_{zw}\|_{H_\infty} < \gamma$*, where* A *is given by (6.33) and* $A \in GH_\infty$*.*

The compensators K which solve the H_∞ sub-optimal problem will be called *admissible compensators*. A compensator will be called *optimal* if it is admissible and minimize the infinity norm of T_{zw} over all admissible compensators.

6.3 A necessary condition

We begin with a simple observation. Consider Figure 6.1 and a signal $w \in L_2(\mathbb{R})$ for which $y(t)$ is zero for all $t < 0$. For this signal the feedback input u will be zero for negative times for every causal controller K. This implies a necessary condition for the existence of a causal, stabilizing γ sub-optimal controller K as is given in the following lemma.

Lemma 6.19 (Necessary condition) *A necessary condition for the existence of a solution for the standard H_∞ sub-optimal control problem is that the space*

$$B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]} = \left\{ \left[\begin{array}{c} w \\ z_- \end{array}\right] \in H_2^\perp \mid \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] \left[\begin{array}{c} w \\ z_- \end{array}\right] \in H_2 \right\} \tag{6.47}$$

is strictly positive in the $J_{\frac{1}{\gamma},n_w,n_z}$-inner product.

Proof: Suppose that K is a stabilizing sub-optimal compensator with bound γ, where γ is a given nonnegative real number, and let $K = K_d^{-1}K_n$ be a left-coprime factorization of K over H_∞. Let $\left[\begin{array}{c} w \\ z_- \end{array}\right]$ be an arbitrary element in $B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]}$. Since the standard H_∞ sub-optimal control problem has a solution there exists an input signal u such that the equation (6.46) is satisfied.
From the equation (6.46) and from the equality

$$A \left[\begin{array}{c} z_- \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} D_1 \\ 0 \end{array}\right] z_-, \tag{6.48}$$

we obtain

$$A \left[\begin{array}{c} z - z_- \\ y \\ u \end{array}\right] = \left[\begin{array}{cc} N_1 & -D_1 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} w \\ z_- \end{array}\right]. \tag{6.49}$$

Since, by (6.47), the right-hand side of this equality is in $H_2^{n_z+n_y+n_u}$, and since $A \in GH_\infty^{(n_z+n_y+n_u)\times(n_z+n_y+n_u)}$, we obtain that $z - z_-$, u and y are in H_2. Obviously, we can write z uniquely as

$$z = z - z_- + z_-,$$

where $z - z_- \in H_2$ and $z_- \in H_2^\perp$.

We have that the transfer matrix T_{zw} from w to z induced by the frequency domain equation (6.46) satisfies $\|T_{zw}\|_{H_\infty} < \gamma$, provided that A has the structure given in (6.32) and $A \in GH_\infty$. We have that

$$\gamma^2 \|w\|_2^2 \geq \|z\|_2^2 \geq \|z_-\|_2^2,$$

so

$$\gamma^2 \|w\|_2^2 - \|z_-\|_2^2 \geq \gamma^2 \|w\|_2^2 - \|z\|_2^2 \geq \gamma^2 \|w\|_2^2 - \|T_{zw}\|_\infty^2 \|w\|_2^2 = (\gamma^2 - \|T_{zw}\|_\infty^2) \|w\|_2^2.$$

Using again the inequality $\gamma^2 \|w\|_2^2 \geq \|z_-\|_2^2$, we obtain

$$\gamma^2 \|w\|_2^2 - \|z_-\|_2^2 \geq \frac{1}{2}(\gamma^2 - \|T_{zw}\|_{H_\infty}^2)(\|w\|_2^2 + \frac{1}{\gamma^2}\|z_-\|_2^2).$$

Dividing by γ^2 we see that this inequality is equivalent with

$$\|w\|_2^2 - \frac{1}{\gamma^2}\|z_-\|_2^2 \geq \frac{1}{2}\min\{1, \frac{1}{\gamma^2}\}(1 - \frac{1}{\gamma^2}\|T_{zw}\|_\infty^2)(\|w\|_2^2 + \|z_-\|_2^2).$$

Recalling that $\|T_{zw}\|_{H_\infty} < \gamma$, the last inequality means that $B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]}$ is strictly positive in $J_{\frac{1}{\gamma},n_w,n_z}$-inner product. ∎

The necessary condition (6.47) holds provided a J-spectral factorization exists.

Lemma 6.20 *If there exists a bistable matrix W such that*

$$N_1(s)N_1^\sim(s) - \gamma^2 D_1(s)D_1^\sim(s) = W(s)\left[\begin{array}{cc} I_{n_y} & 0 \\ 0 & -I_{n_z} \end{array}\right]W^\sim(s), \tag{6.50}$$

for almost all $s \in \mathbb{C}_0$, and the lower-right $n_z \times n_z$ block M_{22} of the matrix-valued function $M := W^{-1}\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]$ is bistable, then the set defined in (6.47) is strictly positive in the $J_{\frac{1}{\gamma}}$-inner product and thus the necessary condition for the solvability of the standard H_∞ sub-optimal control problem is satisfied.

Proof: Suppose that there exists a $W \in GH_\infty^{(n_y+n_z)\times(n_y+n_z)}$ such that (6.50) is satisfied. Obviously, $M \in H_\infty^{(n_y+n_z)\times(n_w+n_z)}$. Since $W \in GH_\infty$, using Lemma 6.11 we have that

$$B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]} = B_M$$

Partition M as

$$M = \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right],$$

where $M_{11} \in H_\infty^{n_y\times n_w}$, $M_{12} \in H_\infty^{n_y\times n_z}$, $M_{21} \in H_\infty^{n_z\times n_w}$ and $M_{22} \in GH_\infty^{n_z\times n_z}$. By (6.50) M satisfies the conditions 1 and 2 of Lemma 6.8 and thus M is co-J_{γ,n_w,n_z}-lossless, i.e.

$$M(s)J_{\gamma,n_w,n_z}M(s)^* \leq J_{n_y,n_z},$$

for almost all $s \in \overline{\mathbb{C}_+}$. The lower-right $n_z \times n_z$ block element of the above inequality is:

$$M_{21}(s)M_{21}(s)^* - \gamma^2 M_{22}(s)M_{22}(s)^* \leq -I_{n_z} < 0,$$

for almost all $s \in \overline{\mathbb{C}_+}$. Since $M_{22} \in GH_\infty^{n_z\times n_z}$, it follows that $H = M_{22}^{-1}M_{21}$ has the H_∞-norm strictly less than γ. So by Lemma 6.12 we obtain that $B_{[M_{21}\ M_{22}]}$ is strictly positive with respect to the $J_{\frac{1}{\gamma},q,p}$ inner product. Obviously, $B_M \subset B_{[M_{21}\ M_{22}]}$ and since $B_{[-N_1\ D_1]} = B_M$, the conclusion follows. ∎

6.4 A two-block problem

In the next section we show that if there exists a bistable matrix-valued function W such that (6.50) holds, then the standard H_∞ sub-optimal control problem is solvable if and only if a related two-block problem is solvable. In this section we formulate and solve a sub-optimal two-block problem.

Definition 6.21 (A two-block problem) *Let $L \in H_\infty^{(n_z+n_w)\times(n_u+n_y)}$, $(n_y = n_w)$ be given and consider the equality*

$$\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} \tag{6.51}$$

for some $\tilde{K}_d \in H_\infty^{n_y\times n_y}$, $\tilde{K}_n \in H_\infty^{n_u\times n_y}$. The sub-optimal two-block problem is to find a $K \in F_\infty$ with the right-coprime factorization $K = \tilde{K}_n\tilde{K}_d^{-1}$ over H_∞ such that

1. *F_2 is bistable and*
2. *$\left\|F_1F_2^{-1}\right\|_{H_\infty} < 1$.*

Lemma 6.22 *Let L be given by*

$$L = W^{-1}[D_2 \;\; -N_2], \tag{6.52}$$

where W is a bistable matrix-valued function, and

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} L \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \tag{6.53}$$

If there exists a bistable matrix V such that

$$R^\sim(s)J_{n_w,n_z}R(s) = V^\sim(s)J_{n_y,n_u}V(s), \tag{6.54}$$

for almost all $s \in \mathbb{C}_0$, and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, then two-block problem is solvable and the set of all K which solves the two-block problem is given by $K = K_nK_d^{-1}$, where

$$\begin{bmatrix} K_n \\ K_d \end{bmatrix} = V^{-1} \begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \tag{6.55}$$

with $U \in H_\infty$ such that $\|U\|_{H_\infty} < 1$.

Proof: Suppose that there exists $V \in GH_\infty$ such that (6.54) is satisfied, and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable. We consider the following equality

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = R \begin{bmatrix} K_n \\ K_d \end{bmatrix}. \tag{6.56}$$

where K_n and K_d are given by (6.55). We have that

$$\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} K_n \\ K_d \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} VV^{-1} \begin{bmatrix} U \\ I \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} = I$$

which means that K_n and K_d are right-coprime. Consider the controller $K = K_n K_d^{-1}$, then

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = RV^{-1} \begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \tag{6.57}$$

where $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$. Lemma 4.4 shows that F_2 is bistable and $\|F_1 F_2^{-1}\|_{H_\infty} < 1$. So the two-block problem is solved.

It remains to prove that for every K which solves the two-block problem there exists a right-coprime factorization $K = K_n K_d^{-1}$ such that K_n and K_d have the form given by (6.55), with $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$.

Consider a K which solves the two-block problem, with the right-coprime factorization $K = X_n X_d^{-1}$. From (6.56) we have that

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = R \begin{bmatrix} X_n \\ X_d \end{bmatrix} = (RV^{-1})V \begin{bmatrix} X_n \\ X_d \end{bmatrix} = RV^{-1} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{6.58}$$

where

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = V \begin{bmatrix} X_n \\ X_d \end{bmatrix}. \tag{6.59}$$

Since K solves the two-block problem, F_2 is bistable and $\|F_1 F_2^{-1}\|_{H_\infty} < 1$. Using also the assumption that the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, we can apply Lemma 4.4 and obtain that Q_2 is a bistable matrix and $\|Q_1 Q_2^{-1}\|_{H_\infty} < 1$.

We have that

$$\begin{bmatrix} X_n \\ X_d \end{bmatrix} = V^{-1} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = V^{-1} \begin{bmatrix} Q_1 Q_2^{-1} \\ I \end{bmatrix} Q_2$$

and since Q_2 is bistable, $K = K_n K_d^{-1}$, where $K_n = X_n Q_2^{-1}$ and $K_d = X_d Q_2^{-1}$, is another right-coprime factorization of the controller. If we denote now $U = Q_1 Q_2^{-1}$, we see that K_n and K_d satisfy (6.55), $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$. ■

6.5 H_∞ control for the Nevanlinna class

In this section we show that the standard H_∞ sub-optimal control problem, as reformulated in Definition 6.18, can be solved, provided that two J-spectral factorizations have a solution.

In the first theorem of this section we show that if the J-spectral factorization (6.50) is solvable, then the H_∞ sub-optimal control problem is solvable if and only if a related two-block problem is solvable.

Theorem 6.23 (Reduction to a two-block problem) *Consider the standard H_∞ sub-optimal control problem of Definition 6.18. Assume that there exists a bistable matrix W such that (6.50) holds, with the lower right $n_z \times n_z$ block of the matrix $W^{-1}[-N_1\ D_1]$ bistable. Then there exists a stabilizing controller $K \in F_\infty$ for the system (6.29) solving the standard H_∞ sub-optimal control problem if and only if the two-block problem of Definition 6.21, with*

$$L = W^{-1}\begin{bmatrix} D_2 & -N_2 \end{bmatrix} \tag{6.60}$$

is solvable. Moreover, the solutions for both problems coincide.

Proof: The proof will be given in four steps.

In step 1 we obtain the equivalent condition for the stability of the closed-loop system. In step 2 a similar result is obtained but now for a system related to the two-block problem. In the last two steps the necessary and sufficient condition of the theorem is proved.

Step 1: Let $K \in F_\infty$ be any stabilizing controller for the plant (6.29), and let $K = K_d^{-1}K_n$ be a left-coprime factorization over H_∞. The closed-loop system is given by

$$\begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} w \\ z \\ y \\ u \end{bmatrix} = 0. \tag{6.61}$$

Denote

$$\Omega := \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}. \tag{6.62}$$

From Theorem 2.35 we know that K has a right coprime factorization over H_∞, i.e. $K = \tilde{K}_n\tilde{K}_d^{-1}$. Furthermore, by Curtain and Zwart [23] Lemma A.7.44, page 661, there exists a matrix-valued function U invertible over H_∞ of the form

$$U = \begin{bmatrix} \tilde{K}_d & * \\ \tilde{K}_n & * \end{bmatrix}. \tag{6.63}$$

such that

$$\begin{bmatrix} -K_n & K_d \end{bmatrix} U = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix}. \tag{6.64}$$

Defining the signals l_1 and l_2 using this U, via

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, \tag{6.65}$$

we obtain the following equivalent representation for the system

$$\begin{bmatrix} -N_1 & D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} w \\ z \\ l_1 \\ l_2 \end{bmatrix} = 0 \tag{6.66}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}. \tag{6.67}$$

However $I_{n_u} l_2 = 0$ is the same as $l_2 = 0$, and this representation becomes

$$\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \begin{bmatrix} z \\ l_1 \end{bmatrix} = N_1 w \tag{6.68}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} l_1 \tag{6.69}$$

$$l_2 = 0. \tag{6.70}$$

Since

$$\begin{aligned} \Omega \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} &= \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = & (6.71) \\ &= \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & I_{n_u} \end{bmatrix} & (6.72) \end{aligned}$$

and $U \in GH_\infty$, we have the following equivalence

$$\Omega \in GH_\infty \text{ if and only if } \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \in GH_\infty. \tag{6.73}$$

Step 2: Using the bistable matrix-valued function W which satisfies relation (6.50) we define

$$\begin{bmatrix} -\tilde{N}_1 & \tilde{D}_1 & \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} = W^{-1} \begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \end{bmatrix}. \tag{6.74}$$

Furthermore, for the plant

$$P = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} & \tilde{N}_2 \end{bmatrix}, \tag{6.75}$$

we define the closed-loop system (see (6.28))

$$\begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & 0 & K_n & K_d \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \tilde{z} \\ y \\ u \end{bmatrix} = 0, \tag{6.76}$$

where K is a controller of the form $K = K_d^{-1} K_n$, $\tilde{w}$ is the new exogenous input, and the new to be controlled output is $\tilde{z}$.
We define the matrices $\tilde{F}_1$ and $\tilde{F}_2$ via

$$\begin{bmatrix} \tilde{F}_2 \\ \tilde{F}_1 \end{bmatrix} = \begin{bmatrix} \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix}, \tag{6.77}$$

where $\tilde{K}_n$ and $\tilde{K}_d$ are given in (6.63).
Denote

$$\tilde{\Omega} := \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & -K_n & K_d \end{bmatrix}. \tag{6.78}$$

Then, for U of Step 1 (see (6.63)),

$$\begin{aligned} \tilde{\Omega}\begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & -K_n & K_d \end{bmatrix}\begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} \quad (6.79)\\ &= \begin{bmatrix} 0 & \tilde{F}_2 & * \\ I_{n_z} & \tilde{F}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix}. \quad (6.80) \end{aligned}$$

Since $U \in GH_\infty$, the following equivalence holds

$$\tilde{\Omega} \in GH_\infty \text{ if and only if } \tilde{F}_2 \in GH_\infty. \tag{6.81}$$

Using

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = U^{-1}\begin{bmatrix} y \\ u \end{bmatrix} \tag{6.82}$$

and (6.79), (6.80), we obtain the equivalent representation for the new system (6.76)

$$\begin{bmatrix} 0 & \tilde{F}_2 & * \\ I_{n_z} & \tilde{F}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix}\begin{bmatrix} \tilde{z} \\ l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -I_{n_w} \\ 0 \\ 0 \end{bmatrix}\tilde{w} \tag{6.83}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}. \tag{6.84}$$

Suppose that the controller K stabilizes the closed-loop system (6.76). Using Lemma 6.17 we have that $\tilde{\Omega} \in GH_\infty$, and thus $\tilde{F}_2 \in GH_\infty$, see (6.81). Consequently, we can write

$$\begin{aligned} l_2 &= 0, \\ l_1 &= -\tilde{F}_2^{-1}\tilde{w}, \\ \tilde{z} &= -\tilde{F}_1 l_1 \end{aligned}$$

which gives us the transfer function from $\tilde{w}$ to $\tilde{z}$, namely

$$T_{\tilde{z}\tilde{w}} = \tilde{F}_1\tilde{F}_2^{-1}. \tag{6.85}$$

Step 3: In this step we show that if the standard H_∞ sub-optimal control problem is solvable, then the two-block problem is solvable. Suppose that the system (6.29) is stabilized by some controller $K \in F_\infty$ with $K = K_d^{-1}K_n$ a left-coprime factorization over H_∞, such that $\|T_{zw}\|_{H_\infty} < \gamma$, for some positive real number γ. By Lemma 6.17 and the equivalence (6.73), this implies that

$$\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \in GH_\infty,$$

where $\tilde{K}_d$ and $\tilde{K}_n$ are given by (6.63). We will prove that the system (6.76) is stable, which is equivalent (by (6.81)) to $\tilde{F}_2 \in GH_\infty$. Furthermore, we will prove that $\|\tilde{F}_1\tilde{F}_2^{-1}\|_{H_\infty} < 1$. Define

$$E := \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}^{-1}, \tag{6.86}$$

and partition E compatibly with $E = \begin{bmatrix} T \\ V \end{bmatrix}$. Then

$$\begin{bmatrix} T \\ V \end{bmatrix} \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} = \begin{bmatrix} I_{n_z} & 0 \\ 0 & I_{n_y} \end{bmatrix}. \tag{6.87}$$

Multiplying both sides of the system (6.68) from the left by E, and using (6.87), we get

$$\begin{bmatrix} I_{n_z} & 0 \\ 0 & I_{n_y} \end{bmatrix} \begin{bmatrix} z \\ l_1 \end{bmatrix} = \begin{bmatrix} TN_1 \\ VN_1 \end{bmatrix} w. \tag{6.88}$$

For $[H_1 \; H_2]$ defined by

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} := T \begin{bmatrix} -N_1 & D_1 \end{bmatrix}, \tag{6.89}$$

we have from (6.87) that $F_2 = I_{n_z} \in GH_\infty$. From (6.89) and (6.88), we have that

$$\|F_2^{-1}F_1\|_{H_\infty} = \|TN_1\|_{H_\infty} = \|T_{zw}\|_{H_\infty} < \gamma.$$

Denote

$$\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} = TW, \tag{6.90}$$

so

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}. \tag{6.91}$$

Since (6.50) holds, we have that the matrix $X = W^{-1}[-N_1 \; D_1]$ satisfies

$$XJX^{\sim} = J$$

almost everywhere on the imaginary axis. Moreover, we made the assumption that the lower-right block of the matrix X is bistable. So, the conditions from Lemma 6.8 are satisfied. Applying this lemma we obtain that $W^{-1}[-N_1 \; D_1]$ is co-J-lossless. We apply now Lemma 4.5 and conclude that

$$\tilde{T}_2 \in GH_\infty \text{ and } \|\tilde{T}_2^{-1}\tilde{T}_1\|_{H_\infty} < 1.$$

For $\tilde{F}_1$ and $\tilde{F}_2$ as defined in (6.77), that is

$$\begin{bmatrix} \tilde{F}_2 \\ \tilde{F}_1 \end{bmatrix} = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix}$$

we have the following sequence of equalities

$$\begin{aligned}\left[\begin{array}{cc}\tilde{T}_1 & \tilde{T}_2\end{array}\right]\left[\begin{array}{c}\tilde{F}_2\\ \tilde{F}_1\end{array}\right] &\overset{(6.90)}{=} TW\left[\begin{array}{c}\tilde{F}_2\\ \tilde{F}_1\end{array}\right]\\ &\overset{(6.77)}{=} TWW^{-1}\left(D_2\tilde{K}_d - N_2\tilde{K}_n\right)\\ &= T\left(D_2\tilde{K}_d - N_2\tilde{K}_n\right). \end{aligned} \tag{6.92}$$

Now, using (6.87) and (6.92), we have that

$$\left[\begin{array}{cc}\tilde{T}_1 & \tilde{T}_2\end{array}\right]\left[\begin{array}{c}\tilde{F}_2\\ \tilde{F}_1\end{array}\right] = 0,$$

which is equivalent to

$$\tilde{T}_1\tilde{F}_2 + \tilde{T}_2\tilde{F}_1 = 0. \tag{6.93}$$

Using again relation (6.87) and also (6.77) gives that

$$\left[\begin{array}{cc}\tilde{T}_1 & \tilde{T}_2\\ I & 0\end{array}\right] W^{-1}\left[\begin{array}{cc} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n\end{array}\right] = \left[\begin{array}{cc} I_{n_z} & 0\\ * & \tilde{F}_2\end{array}\right] \tag{6.94}$$

which, together with $W \in GH_\infty$ and $\left[\begin{array}{cc} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n\end{array}\right] \in GH_\infty$, implies the equivalence

$$\tilde{T}_2 \in GH_\infty \text{ if and only if } \tilde{F}_2 \in GH_\infty. \tag{6.95}$$

Since $\tilde{T}_2 \in GH_\infty$ we get $\tilde{F}_2 \in GH_\infty$. Now, using (6.93) we obtain

$$\tilde{F}_1\tilde{F}_2^{-1} = -\tilde{T}_2^{-1}\tilde{T}_1, \tag{6.96}$$

so

$$\|\tilde{F}_1\tilde{F}_2^{-1}\|_{H_\infty} < 1.$$

Step 4: In this step we show that if the two-block problem is solvable, then the H_∞ sub-optimal control problem is solvable. Using the notation of (6.77) we know that $\tilde{F}_2 \in GH_\infty$ and $\|\tilde{F}_1\tilde{F}_2^{-1}\|_{H_\infty} < 1$. Furthermore, let $K = K_d^{-1}K_n$ be a left-coprime factorization of the controller $K = \tilde{K}_n\tilde{K}_d^{-1}$ which solves the two-block problem. Using step 2 and Lemma 6.17 we see that $K = K_d^{-1}K_n$ is a stabilizing controller for the system (6.76) and that $\|T_{\tilde{z}\tilde{w}}\|_{H_\infty} < 1$. We have to prove that $\left[\begin{array}{cc} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n\end{array}\right] \in GH_\infty$ and $\|T_{zw}\|_{H_\infty} \le \gamma$. Define

$$\left[\begin{array}{cc} F_1 & F_2\end{array}\right] := \left[\begin{array}{cc} -T_{\tilde{z}\tilde{w}} & I\end{array}\right] W^{-1}\left[\begin{array}{cc} -N_1 & D_1\end{array}\right]. \tag{6.97}$$

Using the fact that the matrix $W^{-1}\left[\begin{array}{cc} -N_1 & D_1\end{array}\right]$ is co-J-lossless and the assumption that $\|T_{\tilde{z}\tilde{w}}\| < 1$ we can apply Lemma 4.5 and obtain that

$$\|F_2^{-1}F_1\|_{H_\infty} < \gamma \text{ and } F_2 \in GH_\infty.$$

From (6.97) we see that $\|T_{zw}\|_{H_\infty} = \|F_2^{-1}F_1\|_{H_\infty}$. Also from (6.97) we have that

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I \end{bmatrix} W^{-1} D_1 = F_2 \tag{6.98}$$

and using (6.77) it follows that

$$\begin{bmatrix} I & 0 \end{bmatrix} W^{-1}(D_2\tilde{K}_d - N_2\tilde{K}_n) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_2 \\ \tilde{F}_1 \end{bmatrix} = \tilde{F}_2. \tag{6.99}$$

Combining (6.98) and (6.99) and the fact that

$$\begin{aligned} \begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I_{n_z} \end{bmatrix} W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} &\overset{(6.77)}{=} \begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I_{n_z} \end{bmatrix} \begin{bmatrix} \tilde{F}_2 \\ \tilde{F}_1 \end{bmatrix} \\ &\overset{(6.85)}{=} -\tilde{F}_1\tilde{F}_2^{-1}\tilde{F}_2 + \tilde{F}_1 = 0, \end{aligned}$$

the following equality holds

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I_{n_z} \\ I_{n_w} & 0 \end{bmatrix} W^{-1} \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} = \begin{bmatrix} F_2 & 0 \\ * & \tilde{F}_2 \end{bmatrix}. \tag{6.100}$$

Since F_2, $\tilde{F}_2$ and W are elements of GH_∞, we have that $\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}$ is in GH_∞. Using the step 1 we conclude that the standard H_∞ sub-optimal control problem is solved. ∎

We summarize the results in the following theorem.

Theorem 6.24 *Consider the standard H_∞ sub-optimal control problem in the form (6.28). If there exist bistable matrices W and V such that (6.50) and (6.54) hold, with the lower-right $n_z \times n_z$ block of the matrix $W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ bistable and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} bistable, then the H_∞ sub-optimal control problem is solvable and the set of all stabilizing controllers is given by (*6.55*).*

Remark 6.25 *Using Lemma 6.8, we see that the above theorem states that if there exist bistable matrices W and V such that $W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ is co-J-lossless and the matrix-valued function RV^{-1} is J-lossless, then the standard H_∞ sub-optimal control problem is solvable, and we have a formula for all stabilizing controllers for this H_∞ control problem.*

Remark 6.26 *For rational transfer functions, the existence of a stabilizing controller implies the existence of bistable transfer matrices W and V such that (6.50) and (6.54) are satisfied. However, for our class of transfer functions this no longer holds. We refer to Staffans, Ball and Spitkowski for a counterexample. In the following section we prove that, for transfer functions in a quotient field of the Wiener algebra, the above result is an if and only if result.*

We can write, similar to the rational case (see Meinsma [49], page 64), the following algorithm.

Algorithm 6.27 *Let us consider a stabilizable plant $G \in F_\infty$.*

Step 1: *Find a left-coprime factorization of the plant*

$$G = D^{-1}N$$

as in (6.25), with $D = [D_1\ D_2]$ and $N = [N_1\ N_2]$ corresponds to the partitioning of the output and the input of G.

Step 2: *Choose a real strictly positive γ.*

Step 3: *Compute, if possible, the matrix $W \in GH_\infty$ such that (6.50) holds. If this solution exists and if the lower-right $n_z \times n_z$ block of the matrix-valued function $W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ is bistable, then proceed with the next step.*

Step 4: *Compute, if possible, the matrix $V \in GH_\infty$ such that (6.54) is satisfied.*

If this solution exists and if the lower-right $n_{n_y} \times n_{n_y}$ block of the matrix RV^{-1} is bistable, then proceed with the next step.

Step 5: *Choose an arbitrary $U \in H_\infty$ of appropriate size such that $\|U\|_{H_\infty} < 1$. A stabilizing controller for the standard H_∞ sub-optimal control problem is given by $K = K_n K_d^{-1}$, where K_n and K_d are given by (6.55).*

6.6 H_∞ control for two classes of infinite-dimensional systems

In the previous section we proved that if two J-spectral factorizations are solvable, then the standard H_∞ sub-optimal control problem is solvable. For the two classes of transfer functions ($\hat{\mathcal{B}}_0$ and $\hat{\mathcal{B}}_-$) considered, we show that this result is necessary and sufficient. Hence we need to show that the solvability of the standard H_∞ sub-optimal control problem implies the existence of the two J-spectral factorizations. The main tool in proving the existence of the J-spectral factorizations is the notion of equalizing vectors introduced in Chapter 3. We will formulate all the results for $\hat{\mathcal{B}}_-$. For $\hat{\mathcal{B}}_0$ similar results can be formulated and then proved in an analogous manner.

Let $G \in \hat{\mathcal{B}}_-^{n\times m}$ be a stabilizable plant and let

$$G = [D_1\ D_2]^{-1}\,[N_1\ N_2]$$

be a left-coprime factorization of G over $\hat{\mathcal{A}}_-$, where $[D_1\ D_2]$ and $[N_1\ N_2]$ are partitioned with respect to the outputs and inputs, respectively. This factorization exists since $G \in \hat{\mathcal{B}}_-$ (see Chapter 2, Section 2.3). Since $\hat{\mathcal{A}}_- \subset \hat{\mathcal{A}} \subset H_\infty$, we may call the elements of $\hat{\mathcal{A}}_-$ *stable* matrix-valued functions. Moreover, a matrix-valued function is called *bistable* if it is stable, its inverse exists and it is also stable. The following theorem states the main result of this section.

Theorem 6.28 *The standard H_∞ sub-optimal control problem has a solution $K \in \hat{\mathcal{B}}_-$ if and only if there exist bistable matrix-valued function W and V such that*

$$N_1(s)N_1^\sim(s) - \gamma^2 D_1(s)D_1^\sim(s) = W(s)J_{n_y,n_z}W^\sim(s), \tag{6.101}$$

for all $s \in \mathbb{C}_0$ with the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ bistable, and

$$R^\sim(s)J_{n_w,n_z}R(s) = V^\sim(s)J_{n_y,n_u}V(s) \tag{6.102}$$

for all $s \in \mathbb{C}_0$, where

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} W^{-1}[-N_2\ D_2] \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \tag{6.103}$$

and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable. Moreover, the set of all stabilizing controllers $K \in \hat{\mathcal{B}}_-$ is given by $K = K_d^{-1}K_n$, where

$$\begin{bmatrix} K_n \\ K_d \end{bmatrix} = V^{-1}\begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \tag{6.104}$$

with $U \in \hat{\mathcal{A}}_-$ such that $\|U\|_{H_\infty} \leq 1$.

Similarly as in Definition 3.3 and Definition 3.7 we introduce the notions of co-J-spectral factorization and co-equalizing vector.

Definition 6.29 *Let $Z = Z^\sim \in \hat{\mathcal{W}}$ be a matrix-valued function. We say that Z has a* co-J-spectral factorization *if there exists a matrix-valued function $W \in G\hat{\mathcal{A}}$ such that*

$$Z(s) = W(s)JW^\sim(s) \text{ for all } s \in \mathbb{C}_0 \cup \{\infty\}. \tag{6.105}$$

A solution W of (6.105) is called a co-J-spectral factor *of the matrix-valued function Z.*

Definition 6.30 *A vector u is a* co-equalizing vector *of the matrix-valued function $Z \in \hat{\mathcal{W}}$ if u is a nonzero element of $H_2^\perp$ with Zu in H_2.*

The following theorem can be proved similarly as Theorem 3.9.

Theorem 6.31 *Let $Z = Z^\sim \in \hat{\mathcal{W}}$ be a matrix-valued function such that* $\det Z(s) \neq 0$*, for all $s \in \mathbb{C}_0 \cup \{\infty\}$. The following statements are equivalent*

1. *Z admits a co-J-spectral factorization;*
2. *Z has no equalizing vectors.*

In case $Z = Z^\sim \in \hat{\mathcal{W}}_- \subset \hat{\mathcal{W}}$, it can be proved that the J-spectral factor is invertible over $\hat{\mathcal{A}}_-$, so it is a bistable matrix-valued function. (see Bart et al. [9]).

The following lemma provides the existence of the bistable matrix W such that (6.101) is satisfied.

Lemma 6.32 *Assume that* $\det(N_1 N_1^\sim - D_1 D_1^\sim) \neq 0$ *on* $\mathbb{C}_0 \cup \{\infty\}$*. If the space*

$$B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]} = \left\{ \left[\begin{array}{c} w \\ z \end{array}\right] \in H_2^\perp \mid \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] \left[\begin{array}{c} w \\ z \end{array}\right] \in H_2 \right\} \tag{6.106}$$

is strictly positive in the $J_{\frac{1}{\gamma},n_w,n_z}$*-inner product, then there exists a bistable matrix-valued function* W *such that the equality (6.101) is satisfied.*

Proof: We prove, by contradiction, that there is no vector $v \in H_2^\perp$ such that

$$\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \in H_2. \tag{6.107}$$

Thus we prove that $\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]$ has no co-equalizing vectors. Suppose that $v \in H_2^\perp$ is a nonzero vector satisfying (6.107). Since $[-N_1 \; D_1] \in \hat{\mathcal{A}}$ we have that

$$u = J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \in B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]}.$$

On the other hand

$$\begin{aligned} \langle u, Ju \rangle &= \langle J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v, JJ \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \rangle \\ &= \langle \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v, J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \rangle \\ &= \langle v, \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \rangle = 0 \end{aligned}$$

since $v \in H_2^\perp$ and $\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right] J \left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]^\sim v \in H_2$. This contradicts the strict positivity of the space $B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]}$.
Using Theorem 6.31 we have that there exists a bistable matrix-valued function W such that the equality (6.101) is satisfied. ■

A transposed version of Lemma 4.4 can be proved in the same way as in the proof of Lemma 4.5.

Lemma 6.33 *Let* $M \in \hat{\mathcal{A}}_-^{(n_y+n_z)\times(n_w+n_z)}$ *with* $n_y = n_w$*, and suppose that*

$$M(s) J_{\gamma,n_y,n_z} M^\sim(s) = J_{n_w,n_z}, \text{ for all } s \in \mathbb{C}_0 \cup \{\infty\}. \tag{6.108}$$

Consider the equality

$$\left[\begin{array}{cc} F_1 & F_2 \end{array}\right] = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right] \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right] \tag{6.109}$$

with $F_1 \in \hat{\mathcal{A}}_-^{n_z\times n_w}$, $F_2 \in \hat{\mathcal{A}}_-^{n_z\times n_z}$, $U_1 \in \hat{\mathcal{A}}_-^{n_z\times n_y}$, $U_2 \in \hat{\mathcal{A}}_-^{n_z\times n_z}$, $M_{11} \in \hat{\mathcal{A}}_-^{n_y\times n_w}$, $M_{12} \in \hat{\mathcal{A}}_-^{n_y\times n_z}$, $M_{21} \in \hat{\mathcal{A}}_-^{n_z\times n_w}$ *and* $M_{22} \in \hat{\mathcal{A}}_-^{n_z\times n_z}$*. Then the following two conditions are equivalent*

1. F_2 *is bistable and* $\left\|F_2^{-1}F_1\right\|_{H_\infty} < \gamma$.
2. M_{22} *and* U_2 *are bistable and* $\left\|U_2^{-1}U_1\right\|_{H_\infty} < 1$.

We formulate the sub-optimal two-block problem, and give sufficient conditions such that the standard sub-optimal H_∞ control problem to be reduced to the sub-optimal two-block problem.

Definition 6.34 *Let $L \in \hat{\mathcal{A}}_-^{(n_z+n_w)\times(n_u+n_y)}, (n_y = n_w)$ be given and consider the equality*

$$\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} \tag{6.110}$$

for some $\tilde{K}_d \in \hat{\mathcal{A}}_-^{n_y\times n_y}$, $\tilde{K}_n \in \hat{\mathcal{A}}_-^{n_u\times n_y}$. The sub-optimal two-block problem *is to find a $K \in \hat{\mathcal{B}}_0$ with the right-coprime factorization $K = \tilde{K}_n\tilde{K}_d^{-1}$ over $\hat{\mathcal{A}}_-$ such that*

1. *F_2 is bistable and*
2. *$\left\|F_1F_2^{-1}\right\|_{H_\infty} < 1$.*

As in Lemma 6.17, one can prove that the definition of the standard H_∞ sub-optimal control problem can be reformulated as follows.

Definition 6.35 *Given a transfer matrix*

$$\begin{bmatrix} -N_1 & D_1 & -N_2 & D_2 \end{bmatrix} \in \hat{\mathcal{A}}_-^{(n_z+n_y)\times(n_w+n_z+n_u+n_y)},$$

find a compensator $K \in \hat{\mathcal{B}}_-^{n_u\times n_y}$ with a left-coprime factorization $K_d^{-1}K_n$, such that the transfer matrix T_{zw} (see Figure 6.1) from w to z induced by the frequency domain equation

$$A \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} w \tag{6.111}$$

satisfies $\|T_{zw}\|_{H_\infty} < \gamma$, where A is given by

$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}. \tag{6.112}$$

and it is bistable.

Theorem 6.36 *Consider the standard H_∞ sub-optimal control problem of Definition 6.35. Assume that there exists a bistable matrix-valued function W such that (6.101) holds, with the lower right $n_z \times n_z$ block of the matrix $W^{-1}[-N_1\ D_1]$ bistable. Then there exists a stabilizing controller $K \in \hat{\mathcal{B}}_-$ for the given plant solving the standard H_∞ sub-optimal control problem if and only if the two-block problem, with*

$$L = W^{-1}\begin{bmatrix} D_2 & -N_2 \end{bmatrix} \tag{6.113}$$

has a solution. Moreover, the solutions for both problems coincides.

Proof: The proof follows the same lines as in the proof of Theorem 6.23. ■

Sufficient conditions for solving the two-block problem are given in the following lemma.

Lemma 6.37 *Let L be*

$$L = W^{-1}[D_2 \ \ -N_2], \tag{6.114}$$

and

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} L \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \tag{6.115}$$

If there exists a bistable matrix-valued function V such that

$$R^{\sim}(s) J_{n_w,n_z} R(s) = V^{\sim}(s) J_{n_y,n_u} V(s) \text{ for all } s \in \mathbb{C}_0, \tag{6.116}$$

and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, then the sub-optimal two-block problem is solvable, and the set of all solutions $K \in \hat{\mathcal{B}}_-$ is given by $K = K_n K_d^{-1}$, where K_n and K_d satisfy (6.55) with $U \in \hat{\mathcal{A}}_-$ such that $\|U\|_{H_\infty} < 1$.

Proof: The proof follows the same lines as in the proof of Lemma 6.22. The fact that $K \in \hat{\mathcal{B}}_-$ can be proved similarly as for Theorem 5.12 (see the implication $2 \Rightarrow 1$,a). ■

The following necessary condition should hold for the solvability of the sub-optimal two-block problem.

Lemma 6.38 *A necessary condition for the sub-optimal two-block problem to have a solution is that the equality (6.116) holds for some bistable matrix-valued function V.*

Proof: Let L be given by (6.114) and suppose that the sub-optimal two-block problem has a solution $K = K_n K_d^{-1} \in \hat{\mathcal{B}}_-$. From the equality (6.110) we have that

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = R \begin{bmatrix} \tilde{K}_n \\ \tilde{K}_d \end{bmatrix}. \tag{6.117}$$

From Lemma 6.12 we have that the set generated by $\begin{bmatrix} F_1^T & F_2^T \end{bmatrix}$ is strictly positive in the J_{n_z,n_y}-inner product. Using the equality (6.117), we obtain that

$$B_{\begin{bmatrix} F_1^T & F_2^T \end{bmatrix}} = B_{\begin{bmatrix} \tilde{K}_n \\ \tilde{K}_d \end{bmatrix}^T R^T},$$

so the space $B_{\begin{bmatrix} \tilde{K}_n \\ \tilde{K}_d \end{bmatrix}^T R^T}$ is strictly positive in the J_{n_z,n_y}-inner product. Since the following inclusions hold,

$$\begin{bmatrix} \tilde{K}_n \\ \tilde{K}_d \end{bmatrix}^T R^T B_{R^T} \subset \begin{bmatrix} \tilde{K}_n \\ \tilde{K}_d \end{bmatrix}^T H_2 \subset H_2,$$

we have that

$$B_{R^T} \subset B_{\left[\begin{array}{c} \tilde{K}_n \\ \tilde{K}_d \end{array}\right]^T R^T}.$$

Due to the fact that B_{R^T} is a subset of a strict positive subspace it is strict positive itself. In a similar way as in the proof of Lemma 6.32, it can be shown that there exists a bistable matrix-valued function V^T such that the transposed version of the equality (6.116) is satisfied. It results that V satisfies (6.116). ■

Proof of Theorem 6.28:
The sufficiency follows from Theorem 6.36 and Theorem 6.37. For the necessity we see from Lemma 6.19 that strict positivity of the space $B_{\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]}$ is a necessary condition. Applying now Lemma 6.32 we have that there exists a bistable matrix-valued function W such that the equality (6.101) is satisfied. Since the standard H_∞ sub-optimal control problem is solvable, using Lemma 6.33 and the equality (6.91) we obtain that the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1}\left[\begin{array}{cc} -N_1 & D_1 \end{array}\right]$ is bistable.

Applying Lemma 6.38 we have that the equality (6.54) holds. Since the two-block problem is solvable, from the equality (6.55) and Lemma 4.4 it follows that the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable.

The fact that all solutions have the form (6.104) is a consequence of Lemma 6.22.

6.7 Example

In this section we present an example which illustrates how we can apply the Algorithm 6.27 to a systems with delay.

Consider the plant G given by

$$G = \left[\begin{array}{cc} 0 & -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} \\ \frac{s+1}{s-1} & -\frac{1}{s-1}e^{-\tau s} \end{array}\right]. \tag{6.118}$$

It is easy to see that $G = D^{-1}N$, where

$$\begin{aligned} D &= \left[\begin{array}{cc} 0 & \frac{s-1}{s+1} \\ 1 & 0 \end{array}\right], \\ N &= \left[\begin{array}{cc} 1 & -\frac{1}{s+1}e^{-\tau s} \\ 0 & -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} \end{array}\right], \end{aligned}$$

is a left-coprime factorization of G over $\hat{\mathcal{A}}$. Let $\gamma > 0$ be a given real number. We will find a matrix $W \in G\hat{\mathcal{A}}$ such that the equality

$$N_1 N_1^{\sim} - \gamma^2 D_1 D_1^{\sim} = W J_{1,1} W^{\sim}, \tag{6.119}$$

with

$$\begin{aligned} N_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

holds on the imaginary axis. We compute the left hand side of the equality (6.119)

$$\begin{aligned} N_1 N_1^{\sim} - \gamma^2 D_1 D_1^{\sim} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \gamma^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix}. \end{aligned}$$

If we take the matrix W to be $W = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$, the equality (6.119) holds. The inverse of the matrix W exists and it is given by

$$W^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}.$$

We have that the lower-right element of the matrix

$$W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}.$$

is a constant, so it is bistable.
We will find, using the procedure described by Meinsma ans Zwart [51], a matrix $V \in G\hat{\mathcal{A}}$ such that the relation

$$R^{\sim} J_{1,1} R = V^{\sim} J_{1,1} V \tag{6.120}$$

is satisfied on the imaginary axis, where

$$\begin{aligned} R &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, && (6.121) \\ L &= W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix}, && (6.122) \\ D_2 &= \begin{bmatrix} \frac{s-1}{s+1} \\ 0 \end{bmatrix}, \\ N_2 &= \begin{bmatrix} -\frac{1}{s+1} e^{-\tau s} \\ -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix}. \end{aligned}$$

Replacing W^{-1}, D_2 and N_2 in (6.122), we obtain

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} e^{-\tau s} \\ 0 & \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} e^{-\tau s} \\ 0 & \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix}, \end{aligned}$$

and using the definition of the matrix R given in (6.121) we have that

$$\begin{aligned} R &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1}e^{-\tau s} \\ 0 & \frac{1}{\gamma}\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma}\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} & 0 \\ \frac{1}{s+1}e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix}. \end{aligned} \tag{6.123}$$

We compute now the left hand side of the equality (6.120)

$$\begin{aligned} R^{\sim}J_{1,1}R &= \begin{bmatrix} \frac{1}{\gamma}\frac{s+\sqrt{1+e^{-2\tau}}}{s-1}e^{\tau s} & \frac{1}{-s+1}e^{\tau s} \\ 0 & \frac{s+1}{s-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma}\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} & 0 \\ \frac{1}{s+1}e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma}\frac{s+\sqrt{1+e^{-2\tau}}}{s-1}e^{\tau s} & \frac{1}{s-1}e^{\tau s} \\ 0 & -\frac{s+1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma}\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} & 0 \\ \frac{1}{s+1}e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma^2}\frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} & \frac{1}{s+1}e^{\tau s} \\ -\frac{1}{s-1}e^{-\tau s} & -1 \end{bmatrix}. \end{aligned} \tag{6.124}$$

We write $\frac{1}{s-1}e^{-\tau s}$ as the sum of a stable part and a rational part

$$\frac{1}{s-1}e^{-\tau s} = F_{stab}(s) + F_{rat}(s),$$

where

$$\begin{aligned} F_{stab}(s) &= \frac{e^{-\tau s} - e^{-\tau}}{s-1}, \\ F_{rat}(s) &= \frac{e^{-\tau}}{s-1}. \end{aligned} \tag{6.125}$$

We multiply the matrix $R^{\sim}J_{1,1}R$ to the left with the matrix $\begin{bmatrix} 1 & -F_{stab}^{\sim} \\ 0 & 1 \end{bmatrix}$ and to the right with the matrix $\begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix}$ and choosing $\gamma = 1$ we obtain

$$\begin{aligned} &\begin{bmatrix} 1 & -F_{stab}^{\sim} \\ 0 & 1 \end{bmatrix} R^{\sim}J_{1,1}R \begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 1 & -\frac{e^{\tau s}-e^{-\tau}}{-s-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} & \frac{1}{s+1}e^{\tau s} \\ -\frac{1}{s-1}e^{-\tau s} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{e^{-\tau s}-e^{-\tau}}{s-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} - \frac{e^{\tau s}-e^{-\tau}}{s^2-1}e^{-\tau s} & \frac{1}{s+1}e^{\tau s} - \frac{e^{\tau s}-e^{-\tau}}{s+1} \\ -\frac{1}{s-1}e^{-\tau s} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{e^{-\tau s}-e^{-\tau}}{s-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ -\frac{e^{-\tau}}{s-1} & -1 \end{bmatrix}. \end{aligned}$$

We consider the following matrix function

$$Q = \begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ 0 & \frac{s+\sqrt{1+e^{-2\tau}}}{s+1} \end{bmatrix}. \tag{6.126}$$

The equality

$$\begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ -\frac{e^{-\tau}}{s-1} & -1 \end{bmatrix} = Q^{\sim} J_{1,1} Q \tag{6.127}$$

holds on the imaginary axis.
Since $\det Q \in GH_\infty$, the matrix function Q is bistable. We define the matrix function V to be

$$V = Q \begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix}, \tag{6.128}$$

where F_{stab} and Q are given in (6.125) and (6.126). With this V, the equality (6.120) is satisfied on the imaginary axis.
Explicitely V is given by

$$V = \begin{bmatrix} 1 - \frac{e^{-\tau}(e^{-\tau s} - e^{-\tau})}{s^2-1} & \frac{e^{-\tau}}{s+1} \\ -\frac{(s+\sqrt{a+e^{-2\tau}})(e^{-\tau s} - e^{-\tau})}{s^2-1} & \frac{s+\sqrt{a+e^{-2\tau}}}{s+1} \end{bmatrix}. \tag{6.129}$$

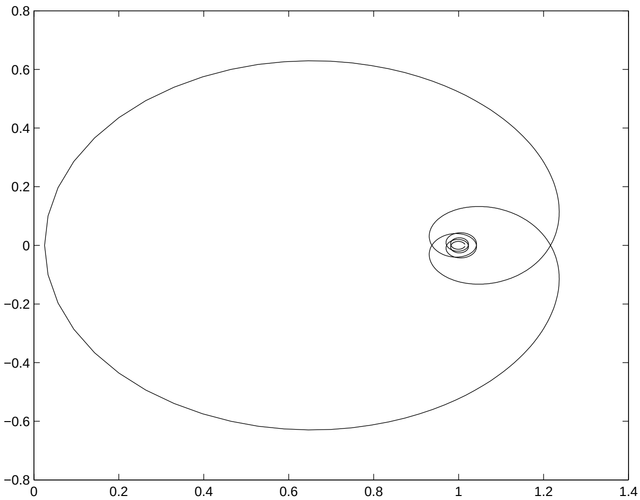

Figure 6.3: The Nyquist plot of $(RV^{-1})_{22}$.

Using (6.123) and (6.129) we obtain that the lower-right element of the matrix-valued function RV^{-1} is

$$(RV^{-1})_{22} = \frac{s + 1 - 2e^{-\tau s}e^{-\tau} + e^{-2\tau}}{s + \sqrt{1 + e^{-2\tau}}}. \tag{6.130}$$

We see in the Figure 6.3 that the Nyquist plot, for $\tau = 0.2$ does not encircle the origin, which means that $(RV^{-1})_{22}$ is bistable.

All the conditions required for applying the algorithm described in Algorithm 6.27 are satisfied, and thus we can construct a H_∞ sub-optimal controller

$$\begin{aligned} K(s) &= F_{stab}(s)\frac{s+\sqrt{1+e^{-2\tau}}-e^{-\tau}}{2s+\sqrt{1+e^{-2\tau}}+1-e^{-\tau}} \\ &= \frac{e^{-\tau s}-e^{-\tau}}{s-1}\,\frac{s+\sqrt{1+e^{-2\tau}}-e^{-\tau}}{2s+\sqrt{1+e^{-2\tau}}+1-e^{-\tau}} \end{aligned}$$

for $\tau = 0.2$.

Chapter 7

The Nehari problem in decomposing R-algebras

Introduction

In this chapter we show that the sub-optimal Nehari problem, as presented in Chapter 3, can be formulated and solved in an algebraic context, without substantial changes to the proofs.

We show that one can obtain a J-spectral factorization in a Banach algebra with identity, provided that it contains the set of rational functions as a dense set and it is continuously embedded in the Banach algebra of continuous functions on the imaginary axis with a well defined limit at infinity. The Wiener algebra on the imaginary axis is an example of such space. The Nehari problem can then be solved, using the existence of a J-spectral factorization.

7.1 Definition and properties of Banach algebras

As before, we denote by $\mathbb{C}$ the set of complex numbers. For the definition of a Banach algebra, we recall first the definition of a Banach space.

Definition 7.1 (Banach space) *A* Banach space *is a complete, normed linear space.*

Definition 7.2 (Banach algebra) *A* Banach algebra *is a complex Banach space B with a multiplicative operation satisfying the following properties:*

1. $(xy)z = x(yz)$,
2. $x(y+z) = xy + xz$, $(y+z)x = yx + zx$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

4. $\|xy\| \leq \|x\|\|y\|$,

where x, y, $z \in B$ *and* $\alpha \in \mathbb{C}$.
If $xy = yx$ *for all* $x, y \in B$, *then* B *is called a* commutative *Banach algebra. An element* $e \in B$ *is the* identity *or* unit *in* B *if* $\|e\| = 1$ *and* $ex = xe = x$ *for all* $x \in B$. *A Banach algebra with a unit is called a* unital *Banach algebra.*

From the properties of the norm, the algebraic operation on B of addition and multiplication are continuous.

The following theorem gives some properties about the invertible elements in a Banach algebra (see Gohberg et al. [30], Theorem XXIX.4.1).

Theorem 7.3 *Suppose that* $x \in B$ *is invertible. If* $\|x - y\| < \|x^{-1}\|^{-1}$, *then* y *is invertible and*

$$y^{-1} = \sum_{k=0}^{\infty} [x^{-1}(x - y)]^k x^{-1},$$

and

$$\|x^{-1} - y^{-1}\| \leq \frac{\|x^{-1}\|^2 \|x - y\|}{1 - \|x^{-1}\|\|x - y\|}.$$

In particular, the set G *of invertible elements in* B *is an open set in* B, *and the map* $x \mapsto x^{-1}$ *is a homeomorphism from* G *onto* G.

Let B be a Banach algebra. By $B^{n\times m}$ we denote the set of all $n \times m$ matrices $[a_{ij}]_{i,j}^m$ with entries in B. We endow $B^{n\times m}$ with the usual matrix operations and the induced norm.

$$\|[a_{ij}]\| := \max_{1\leq i\leq n} \sum_{j=1}^{m} \|a_{ij}\|.$$

With this norm and multiplication, $B^{n\times m}$ is a Banch algebra. If B has a unit e and $n = m$, then $B^{n\times n}$ also has a unit, namely, the $n \times n$ matrix E with e on the diagonal and zeros elsewhere.

In the commutative case the determinant is very useful. For a proof of the following result we refer to Gohberg et al. [30], Theorem XXIX.8.1.

Theorem 7.4 *Let* B *be a unital commutative Banach algebra, and let* $Z \in B^{n\times n}$. *Then* Z *is invertible in* $B^{n\times n}$ *if and only if* $\det(Z)$ *is invertible in* B, *and in that case* Z^{-1} *is given by the Cramer's rule.*

We end this section with the definition of decomposing Banach algebras.

Definition 7.5 (decomposing Banach algebra) *Let* B *be a unital Banach algebra. We call* B *a* decomposing Banach algebra *if* B *has closed subalgebras* B_+ *and* B_-, *both containing non-zero elements, such that*

$$B = B_- \oplus B_+,$$

where by $\oplus$ *we denote the direct sum.*

Let P be the projection of B onto B_+ along B_-, and $Q := I - P$. Using the closed graph theorem, it is easy to see that P and Q are bounded operators.

7.2 Factorization in decomposing Banach algebras

The following abstract result, regarding factorization in decomposing Banach algebras, holds.

Theorem 7.6 *Assume $B = B_- \oplus B_+$ is a decomposing Banach algebra. If $a \in B$ and $\|a\| < \min\{\|P\|^{-1}, \|Q\|^{-1}\}$, then $e - a$ factorizes as*

$$e - a = (e + b_-)(e + b_+)$$

with $e + b_\pm$ invertible in B and

$$b_\pm \in B_\pm, \quad (e + b_\pm)^{-1} - e \in B_\pm.$$

Furthermore, the elements b_+ and b_- are uniquely determined by a and are given by

$$b_\pm := (e + x_\pm)^{-1} - e,$$

where $x_+ \in B_+$ and $x_- \in B_-$ are the unique solutions of the equations

$$x_+ - P(ax_+) = Pa, \quad x_- - Q(x_- a) = Qa.$$

Proof: The proof of this theorem can be found in Gohberg et al. [30], Theorem XXIX.9.1. ■

Now we can continue with Banach algebras of continuous functions. Let C be the Banach algebra of continuous functions on the imaginary axis ($\mathbb{C}_0$) with well defined limit at infinity; i.e., for any $f \in C$,

$$\lim_{|s| \to \infty} f(s)$$

exists. We call $\mathbb{C}_0 \cup \{\infty\}$ the extended imaginary axis. This can be seen as the compactification of the imaginary axis with the point at infinity via the Cayley transform of the unit circle (see also Chapter 3). By $C^{n \times n}$ we denote the Banach algebra of $n \times n$-matrices with entries from C. The norm of an element $Z \in C^{n \times n}$ is defined by

$$\|Z\|_{\mathbb{C}_0} := \sup_{t \in \mathbb{C}_0} \|Z(t)\|.$$

The space $C^{n \times n}$ can be viewed as the space of continuous maps from $\mathbb{C}_0$ into the space $\mathbb{C}^{n \times n}$ of complex-valued $n \times n$-matrices, where the norm of an element in $\mathbb{C}^{n \times n}$ is its norm as an operator on the finite-dimensional space $\mathbb{C}^n$. The group of invertible elements $GC^{n \times n}$ in $C^{n \times n}$ consist of those elements Z satisfying

$$\det Z(s) \neq 0$$

for any s on the extended imaginary axis.

Let B^c be a Banach algebra continuously embedded in C, such that it contains the identity $e(t) = 1$. The norm on B^c will be denoted by $\|\cdot\|_{B^c}$. Define B^c_+ to be the subalgebra consisting of those $f \in B^c$ that have an extension which is continuous on the closed right half-plane, $\overline{\mathbb{C}_+}$, and analytic in the open right half-plane, $\mathbb{C}_+$, and have a well-defined limit at infinity

$$\lim_{\substack{|s|\to\infty \\ \mathrm{Re}(s)\geq 0}} f(s).$$

Let $B^c_{0,-}$ consist of those $f \in B^c$ that have an extension which is continuous on the closed left half-plane, $\overline{\mathbb{C}_-}$, and analytic in the open left half-plane, $\mathbb{C}_-$, and have limit zero at infinity

$$\lim_{\substack{|s|\to\infty \\ \mathrm{Re}(s)\leq 0}} f(s) = 0.$$

B^c_+ and $B^c_{0,-}$ are closed subalgebras of B^c (see Clancey and Gohberg [13], pg. 40). We define now the notion of a decomposing Banach algebra of continuous functions on the extended imaginary axis.

Definition 7.7 (decomposing Banach algebra of continuous functions) *Let B^c be a Banach algebra continuously embedded in C, such that it contains the identity $e(t) = 1$. We say that B^c is a* decomposing Banach algebra of continuous functions *on the extended imaginary axis if*

$$B^c = B^c_{0,-} \oplus B^c_+.$$

We have the following characterizarion of decomposing Banach algebras (see Clancey and Gohberg [13], pg. 40).

Theorem 7.8 *Let B^c be a Banach algebra continuously embedded in C, such that it contains the identity $e(t) = 1$. Then B^c is decomposing if there exists a projection of B^c onto B^c_+ which is a bounded operator on B^c.*

We mention that C is not a decomposing Banach algebra. The fact that

$$C \neq C_{0,-} \oplus C_+$$

is demonstrated by Hoffman (see Hoffman [34], pg. 155).

Note that $B^c_+ \subset H_\infty$, so we can call the elements of B^c_+ *stable* transfer functions. We call an element $f \in B^c$ *bistable* if f is stable, its inverse exits and it is also stable. Similarly as in Chapter 3 and 4, respectively, we can define the Toeplitz and the Hankel operators associated to a transfer function in B^c.

Let $R(\mathbb{C}_0)$ be the algebra of rational functions that are in C. Note that $R(\mathbb{C}_0)$ has a natural direct sum decomposition, but it is not a Banach space (the completeness does not hold). Next we define the standard factorization (known also as Wiener-Hopf factorization) and the J-spectral factorization in decomposing Banach algebra of continuous functions on the imaginary axis.

Definition 7.9 (standard factorization) *Let B^c be a decomposing Banach algebra of continuous functions on the imaginary axis such that $R(\mathbb{C}_0) \subset B^c$. Let $s_{\pm,i}$ be fixed points in $\mathbb{C}_\pm$. An $Z \in GB^{c,n\times n}$ is said to admit a* standard factorization *relative to the imaginary axis in B^c if Z can be decomposed as*

$$Z = Z_- D Z_+,$$

where $Z_\pm \in GB_\pm^{c,n\times n}$ and

$$D(s) = \operatorname{diag}\left[\left(\frac{s - s_{-,1}}{s - s_{+,1}}\right)^{k_1}, \ldots, \left(\frac{s - s_{-,n}}{s - s_{+,n}}\right)^{k_n}\right], \; s \in \mathbb{C}_0, \tag{7.1}$$

with $k_i \in \mathbb{Z}$ and $k_1 \geq \ldots \geq k_n$. The integers k_i are called (the right-) partial indices *of the factorization. In the case $k_1 = \ldots = k_n = 0$, so that,*

$$Z = Z_- Z_+, \tag{7.2}$$

*then Z is said to admit a (right-)*canonical factorization *relative to the imaginary axis.*

Recall that we have made the following notation (see Chapter 2)

$$J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

Definition 7.10 (J-spectral factorization) *Let B^c be a decomposing Banach algebra of continuous functions on the imaginary axis. An $Z \in GB^{c,n\times n}$ is said to admit a J-*spectral factorization *relative to the imaginary axis in B^c if Z can be decomposed as*

$$Z = Z_- J_1 Z_+,$$

where $Z_\pm \in GB_\pm^{c,n\times n}$.

Before we establish results about the existence of the above factorizations, we introduce the notion of an R-algebra.

Definition 7.11 (R-algebra) *A Banach algebra B^c of continuous functions on the extended imaginary axis is called an R-*algebra *if it contains $R(\mathbb{C}_0)$ as a dense set.*

We present the Wiener algebra on the imaginary axis as an example.

Example 7.12 The Wiener algebra on the imaginary axis is a decomposing R-algebra (see Chapter 2).

Recall the following notation from Chapter 2

$$F^\sim(s) := F(-\overline{s})^*$$

where by A^* we denoted the transposed conjugate of the complex matrix-valued function A. We see that if $F \in B_+^c$, then $F^\sim \in B_-^c$.

Using Theorem 7.3, it is possible to prove the following result (see also Gohberg and Krupnik [31], Lemma 8.2, pg. 75).

Lemma 7.13 *Every R-algebra is inverse closed. This means that any $f \in B^c$ satisfying $\det f(s) \neq 0$ for s on the extended imaginary axis is invertible in B^c.*

Moreover, the following result holds.

Lemma 7.14 *The following two assertions are equivalent:*

1. *F is invertible over B_+^c;*
2. *$\det F(s) \neq 0$ for all $s \in \overline{\mathbb{C}_+}$ and*

$$\lim_{\substack{|s|\to\infty \\ \mathrm{Re}(s)\geq 0}} \det(F(s)) \neq 0.$$

A similar result as in Lemma 2.8 can be proved for Banach algebras. However, here we present it only for R-algebras.

Lemma 7.15 *Let B^c be an R-algebra, and consider $a \in B^c$ with $\|a\| < 1$. Then, a stable if and only if $(e+a)^{-1}$ is stable.*

Proof: Suppose that $\|a\| < 1$ and $a \in B_+^c$ (stable). The fact that $(e+a)^{-1}$ is stable can be proved as in Lemma 2.8.

Conversely, suppose that $\|a\| < 1$ and $(e+a)^{-1}$ is stable. Since $R(\mathbb{C}_0)$ is dense in B_+^c, there is a sequence of rational functions $(r_n)_{n\in\mathbb{N}} \subset B_+^c$ which converges to $(e+a)^{-1}$. Using Lemma 7.3, we have that the sequence $a_n := r_n^{-1} - e$ converges to a, so we may assume that $\|a_n\| < 1$. It can be easily checked that

$$(e+a_n)^{-1} = r_n \in B_+^c.$$

We can apply Lemma 2.8 for a_n ($\|a_n\| < 1$) to obtain $a_n \in H_\infty$. Since a_n are rational functions, it follows that $a_n \in B_+^c$. Using the closedness of B_+^c in the $\|\cdot\|_{\mathbb{C}_0}$ norm, and the fact that a_n converges to a, we have that a is stable. ∎

For a proof of the following result, about the existence of a standard factorization in decomposing R-algebra of continuous functions on the imaginary axis, we refer the reader to Clancey and Gohberg [13], Theorem 3.1.

Theorem 7.16 *Let B^c be a decomposing R-algebra of continuous functions on the extended imaginary axis. Then, every element $Z \in GB^{c,n\times n}$ admits a standard factorization in B^c relative to the imaginary axis.*

We recall the notion of an equalizing vector. By H_2 and $H_2^\perp$ we denoted the Hardy spaces (see also Chapter 2).

Definition 7.17 (equalizing vector) *A vector u is an* equalizing vector *of $Z \in B^c$ if u is a nonzero element of H_2 with Zu in $H_2^\perp$.*

The following theorem concerning the existence of a J-spectral factorization can be proved similarly as Theorem 3.9.

Theorem 7.18 *Let B^c be a decomposing R-algebra of continuous functions on the extended imaginary axis. Then, the following three assertions are equivalent:*

1. *$Z \in GB^{c,n\times n}$ admits a J-spectral factorization in B^c relative to the imaginary axis;*
2. *The associated Toeplitz operator is boundedly invertible;*
3. *There are no equalizing vectors.*

7.3 The Nehari problem in decomposing R-algebras

Throughout this section we assume that B^c is a decomposing R-algebra. First we present the reformulation of Lemma 4.1 and Lemma 4.2.

Lemma 7.19 *Let P_{21}, P_{22}, Q_1 and Q_2 be stable ($\in B^c_+$) matrix-valued functions of appropriate dimensions. Suppose that P_{22} and Q_2 are invertible in B^c and that the inequalities*

$$\begin{array}{rcl} \|P_{22}^{-1}P_{21}\|_{\mathbb{C}_0} & \leq & 1, \\ \|Q_1Q_2^{-1}\|_{\mathbb{C}_0} & < & 1, \end{array}$$

are satisfied. Then the following two statements are equivalent

1. *P_{22} and Q_2 are bistable;*
2. *$P_{21}Q_1 + P_{22}Q_2$ is bistable.*

Proof: The proof follows the same lines as in the proof of Lemma 2.10. We only have to use Lemma 7.13 instead of Lemma 2.5. ∎

Lemma 7.20 *Let $P \in B^{c,(n_w+n_z)\times(n_y+n_z)}$, and suppose that*

$$P^{\sim}(s)J_{\gamma,n_w,n_z}P(s) = J_{n_y,n_z} \tag{7.3}$$

on the imaginary axis. Consider the equality

$$\left[\begin{array}{c} X_1 \\ X_2 \end{array}\right] = \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} Q_1 \\ Q_2 \end{array}\right] \tag{7.4}$$

with $X_2 \in B^{c,n_z\times n_z}$, $Q_1 \in B_+^{c,n_y\times n_z}$, $Q_2 \in B_+^{c,n_z\times n_z}$, $P_{21} \in B_+^{c,n_z\times n_y}$, $P_{22} \in B_+^{c,n_z\times n_z}$. Then the following two conditions are equivalent

1. *X_2 is bistable and $\left\|X_1X_2^{-1}\right\|_{\mathbb{C}_0} < \gamma$,*
2. *P_{22} and Q_2 are bistable and $\|Q_1Q_2^{-1}\|_{\mathbb{C}_0} < 1$.*

Proof: The proof follows as in Lemma 4.2, and the fact that B^c is inverse closed (from Lemma 7.13). ∎

The sub-optimal Nehari (extension) problem for decomposing Banach algebras can be formulated as follows:

Definition 7.21 (Nehari problem) *Let $G \in B^c$ be a matrix-valued function G and a $\sigma > 0$, find (if it exists) a stable K such that*

$$\|G + K\|_{\mathbb{C}_0} < \sigma.$$

The following theoren is our main result of this chapter.

Theorem 7.22 *Let G be a matrix-valued function $G \in B^{c,k\times m}$, and σ a positive real number. The following statements are equivalent:*

1. $\|H_G\| < \sigma$.
2. *There exists $K \in B_+^{c,k\times m}$ such that*
$$\|G + K\|_{\mathbb{C}_0} < \sigma. \tag{7.5}$$
3. *There exists a J-spectral factor $\Lambda(s) \in B^{c,(k+m)\times(k+m)}$ for*
$$W(s) = \begin{bmatrix} I_k & 0 \\ G^{\sim}(s) & I_m \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_k & G(s) \\ 0 & I_m \end{bmatrix} \tag{7.6}$$
with $\Lambda_{11}^{-1}(s) \in B_+^{c,k\times k}$

Furthermore, all solutions for the sub-optimal Nehari problem are parametrized by

$$K(s) = X_1(s)X_2(s)^{-1},$$

where

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_m \end{bmatrix}, \tag{7.7}$$

with $Q(s) \in B_+^{c,k\times m}$, $\|Q\|_{\mathbb{C}_0} < 1$.

Proof: Using the previous two lemmas, the proof can be done similarly as for Theorem 4.7. ∎

Note that the Algorithm 3.23 for computing the J-spectral factor can be applied in the framework of decomposing R-algebras.

Remark 7.23 *In this chapter we have formulated and solved the sub-optimal Nehari problem in the framework of decomposing R-algebras of continuous functions on the compactified imaginary axis. There is a strong connection between the above results and the abstract approach taken by Ball, Gohberg and Kaashoek in [4]. Theorem 7.22 can be seen as a consequence of Theorem 2.3.b in [4], formulated and solved in the context of an unital Banach algebra with an involution and a band structure. However, the approach taken in this chapter is more transparent.*

Chapter 8

Conclusions and further research

In this chapter we summarize the contents of the thesis, and give an overview of the most important results. Furthermore, some hints and directions for future research are included.

The main control problems investigated in this thesis are H_∞ control problems. The standard H_∞ control problem (see Figure 8.1) is to find a controller K which stabilizes a given plant G, and which makes the infinity norm of the transfer function from w to z smaller than a given positive real number. It is called standard because it includes many H_∞ control problems, if not all, as special cases. In

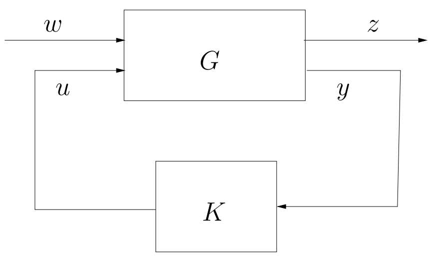

Figure 8.1: The standard H_∞ control problem.

this thesis we propose a J-spectral factorization approach to H_∞ control problems. Using the mixed-sensitivity problem as an example of an H_∞ control problem, we illustrated how the J-spectral factorization arises naturally in H_∞ control (see Chapter 1).

Most of the results in this thesis are presented for the Wiener class of transfer functions or its quotient field. The Wiener class of transfer functions, as introduced in the second chapter, consists of functions for which the corresponding impulse responses are the sum of an integrable function and a delta distribution. The transfer functions of Pritchard-Salamon systems form a particular subclass. The Pritchard-Salamon class of abstract infinite-dimensional state-space systems allows some unboundedness of the control and observation operators. The analytic class, i.e., the class of exponentially stable analytic infinite-dimensional systems, has integrable impulse responses (see Sasane [64], Theorem 3.2.3, page 75). This implies that it is also included in the Wiener class, and so the results obtained in this thesis also apply to the analytic class.

In Chapter 3 we found criteria for the existence of a J-spectral factorization. For the Wiener class of matrix-valued functions, we provide necessary and sufficient conditions for the existence of a J-spectral factorization. One of these conditions states that the existence of a J-spectral factorization is equivalent to the invertibility of an associated Toeplitz operator. The second one states that the J-spectral factorization exists if and only if the to-be-factorized matrix has no equalizing vectors. The equivalence with the invertibility of the Toeplitz operator is known (see M. Weiss [77]). However, the fact that the existence of a J-spectral factorization is equivalent to the non-existence of equalizing vectors was only known for the rational case (see Meinsma [50]).

In the last section of the third chapter, we provide an "algorithm" for computing the J-spectral factor. Although we call this an "algorithm", it is not clear how one can actually write it into a computer program. The main difficulty appears in the second step. There, two algebraic equations (involving projection operators) must be solved in order to proceed with the next step.

For rational matrix-valued functions algorithms for computing the J-spectral factor are well established and are already implemented in the Polynomial Toolbox for MATLAB (see http://www.math.utwente.nl/ssb/ for more information). Since the set of rational functions is a dense subset of our class of infinite-dimensional transfer functions, the algorithm for rational transfer functions may be useful to obtain an J-spectral factor for our class of transfer functions. However, one would need to establish approximation results. Hence, we formulate the problem as follows: Let $Z = Z^{\sim}$ be a matrix-valued function which admits a J-spectral factorization. Find a sequence $(Z_n)_{n\in\mathbb{N}}$ of rational approximates of Z such that $Z_n = Z_n^{\sim}$, and the associated sequence $(V_n)_{n\in\mathbb{N}}$ of J-spectral factors of Z_n converges "reasonably well" to a J-spectral factor of Z. Further research should be done in this direction.

In the Chapters 4-6, we reduce

- the sub-optimal Nehari problem,
- the sub-optimal Hankel norm approximation problem,
- the standard H_∞ sub-optimal control problem,

to J-spectral factorization problem(s). We present now these results chapter by chapter.

In Chapter 4 we give a direct frequency-domain solution for the sub-optimal Nehari extension problem. This is done for the Wiener class of transfer functions.

Curtain and Green stated this result in [15], and they say that this is a special case of the more general results of Ball and Helton [5], [6]. If one looks in the papers [5], [6], the result stated in [15] is not an obvious corollary of the abstract results presented by Ball and Helton. This motivated our investigation. The proofs presented in Chapter 4 are simple, and use standard results from algebra and operator theory. With the result presented in Chapter 4, we fill in the gap between the abstract theory in [5], [6] and the result presented in the paper [15]. In Section 4.4 we formulate the results for Pritchard-Salamon systems, and provide a state-space description of a J-spectral factor.

In Chapter 5, the Hankel norm approximation problem is presented. For the Wiener class of transfer functions, we prove a stronger version of the result obtained by Sasane in [64]. We do this by showing that some of their assumptions always hold. Moreover, we prove that one of their sufficient conditions is also necessary for the solvability of the Hankel norm approximation problem. In Section 5.3 we make some comments regarding the Hankel norm approximation problem for Pritchard-Salamon systems.

In Chapter 6 we reduce the standard H_∞ sub-optimal control problem to two J-spectral factorization problems. This is done for a class of systems with an irrational transfer function. We generalize the approach taken by Meinsma, as presented for the rational case in [49]. The proof are still not very transparent and clear, but even for the rational case it cannot be done simpler (to our knowledges). This is another version of the result found earlier by Curtain and Green [15].

Finally, looking at our proofs, one may see that they are mostly based on algebraic results. This strong relation is made more clear in Chapter 7. We show how one can formulate and solve the sub-optimal Nehari problem in decomposing R-algebras of continuous functions on the compactified imaginary axis. There is a strong connection between these results and the approach taken by Ball, Gohberg and Kaashoek in [4]. We believe that it will be possible to formulate and solve the Hankel norm approximation problem in an algebraic setting, and we recommend it for further research.

Summary

Systems and control theory is an area of research which combines engineering with mathematics. Using mathematical concepts, many engineering problems can be formulated into a mathematical framework and then be solved. This thesis deals with a number of problems arising in H_∞ control for infinite-dimensional systems. The approach for solving this problems is via J-spectral factorization, using frequency-domain techniques.

The H_∞ control problem is, roughly speaking, to find a stabilizing controller for a given plant that minimizes the H_∞ norm of the closed-loop transfer function. By considering the mixed sensitivity problem, which is a particular H_∞ control problem, we illustrate how the J-spectral factorization appears in H_∞ control.

Since the J-spectral factorization plays an important role in H_∞ control, it is important to know when a matrix-valued function possesses such a factorization. For the Wiener class of infinite-dimensional systems (the class of transfer functions for which the corresponding impulse responses are absolutely integrable), we give necessary and sufficient conditions for the existence of a J-spectral factorization.

The so called Nehari problem was introduced for the first time by Z. Nehari in 1957, and it is naturally formulated in the frequency-domain: given a matrix-valued function G, find the distance from G to the stable matrix-valued functions. For the Wiener class of infinite-dimensional systems, we obtain an elementary solution for a version of the Nehari problem, known as the sub-optimal Nehari extension problem. The approach is via J-spectral factorization and it uses the concept of equalizing vectors. The connection between the equalizing vectors, operator theory and the Nehari extension problem is also explained.

Then we consider the Hankel norm approximation problem, which is a generalization of the Nehari problem. This problem has been studied extensively in the literature (see, for example, Adamjan et al. [1], Ball and Ran [7], Glover [28], Ran [59], Glover et al. [29], Curtain and Ran [19], Sasane [64], Sasane and Curtain [65]). We present an elementary derivation of the reduction of the sub-optimal Hankel norm approximation problem to a J-spectral factorization problem. We do this for the Wiener class of infinite-dimensional systems.

In Chapter 6 we consider the standard H_∞ sub-optimal control problem. This

problem is called standard because many of the H_∞ control problems can be reformulated into this problem. For the Nevanlinna class of infinite-dimensional systems (a quotient field of the bounded analytic functions in the open right half-plane), we prove that the standard H_∞ sub-optimal control problem is solvable provided that two J-spectral factorizations have a solution. For the Wiener class of infinite-dimensional systems, we show that the sufficient conditions are also necessary.

In the last chapter of this thesis we give an algebraic formulation for the sub-optimal Nehari extension problem. This formulation has a strong connection with the approach taken by Ball, Gohberg and Kaashoek in [4].

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